Allocation of Marketing Resources to Optimize Customer Equity

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The President:

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Executive summary

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To stay competitive, companies spend billions of dollars each year on building long-term relationships with their customers. Marketing managers are therefore constantly challenged with the problem of how to allocate a limited marketing budget across customers and competing marketing initiatives. This thesis addresses the problem of how to efficiently allocate marketing resources to maximize the financial value generated by marketing investments. Advanced stochastic models addressing the following three issues are proposed: a) maximization of customer lifetime value by linking marketing actions to the financial value generated during the relationship with the company, b) estimation of the financial profile of customers and, c) use of risk management techniques to select the optimal customer portfolio. The combination of these issues has not been addressed by the current marketing literature.

The maximization of customer lifetime value and the evaluation of customers’ financial profiles are addressed in chapter 3. I introduce a methodology, based on cross-validation and customer segmentation algorithms, for the estimation of robust Markov Decision Processes modeling the dynamic relationship between the customers and the company. Using dynamic programming, the optimal marketing policy which maximizes the expected customer lifetime value is found. Customer heterogeneity across states (i.e. customer segments) and marketing action heterogeneity is addressed by tailoring marketing actions to individual customers. To evaluate the financial profile of customers, i.e. the lifetime value distributions, I use Monte Carlo simulation coupled with bootstrap. In this way both the uncertainty in the model parameters and the uncertainty in the customer behavior is captured.
Once the financial profile of customers has been estimated, the use of risk management techniques to select the optimal customer portfolio is addressed in chapter 4. I discuss methods for the allocation of a limited marketing budget by formulating and solving the associated constrained optimization problem. The objective is to maximize expected value while minimizing risk. I show that, unlike assets exchanged in financial markets, when evaluating the risk profile of customer assets there is not always an effective value-risk tradeoff. I discuss the conditions under which a value-risk tradeoff is a relevant issue. For instance, the uncertainty due to the model parameters can have a large impact on the tradeoff. For this reason it is important to consider the parameter uncertainty when estimating the financial profiles of customers.

While in chapters 3 and 4 customer lifetime value is estimated and maximized at a customer segment level, in chapter 5 I introduce an alternative approach for the establishment of a customer portfolio in which Mixtures of Gaussians are used to predict the lifetime value distribution of each customer. As shown in a case study, a portfolio of customers can be created according to financial investment criteria considering the value-risk tradeoff and the risk profile of the decision maker. A risk-neutral criterion maximizes the expected value of the future customer portfolio, but the downside risk is not minimized. On the other hand, a risk-averse criterion leads to portfolios with a lower future expected value but with a lower downside risk as well. The principle of expected utility nicely adapts to mixtures of Gaussians. I derive a closed form for predicting the expected utility of each customer. This allows us to extend and generalize the traditional “rank and cut” practice used to build customer lists for the allocation of marketing resources. The developed approach extends the common practice, which considers only the expected future value, by considering uncertainty. It generalizes the common practice because the expected utility principle leads to the maximization of the expected value in the case of risk neutrality.
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Chapter 1

Introduction

1.1 Problem description and relevance

As markets in developed economies mature, the competitive advantages of economies of scale and scope are reduced. In order to stay competitive, companies need to allocate resources to build long-term relationships with their customers [41]. In fact, today’s companies spend billions of dollars each year to finance marketing activities aimed at managing relationships with existing customers and at acquiring new customers [85]. For instance, the top 16 retailers in Europe spent collectively more than $1 billion in 2000 on loyalty initiatives [73]. As a consequence, marketing managers are constantly challenged with the problem of allocating marketing budgets and selecting the best marketing initiatives [78]. Moreover, top management increasingly requires that marketing contributes to the creation of shareholder value [83].

In this thesis I address the problem of how to efficiently allocate marketing resources to maximize the financial value generated by marketing investments.

As marketing expenditures have a direct impact on shareholder value [22], they can be considered as long-term investments rather than as short-term costs [83]. From this value-based marketing perspective, customers represent intangible assets of the company [5]. Although the value of these assets is not reported in the balance sheet\(^1\), there is strong evidence that the value of customer assets is related to the market value of a firm [33], [34].

In this thesis, I consider the customer equity approach [79], [12], which is based on the customer lifetime value construct [8], [25], to establish a link between marketing investments and their results. Conceptually, the customer lifetime value is defined as the long-term value generated by a customer during his relationship with the company, while the customer

\(^{1}\)It is interesting to note that in some countries, under certain circumstances, the value of intangible brand assets is reported in the balance sheet [6].
equity is the sum of the lifetime values of all the present and future customers [79]. As to the managerial implications, the customer equity paradigm considers customers as financial assets, which like any other assets, should be valued, managed and optimized. By focusing on the long-term relationship with the customers, companies can measure the results of marketing initiatives and select those actions which maximize the customer equity [6], [7], and ultimately the shareholder value [33]. For instance, in a case study analyzed in [34] a 1% improvement of retention rate is found to improve firm value by 5%. According to [8], customer equity management is a dynamic interactive marketing system that uses financial valuation techniques to optimize the acquisition, the retention, and the selling of additional products to the firm’s customers.

Although there is common agreement on the concepts and the managerial implications of customer equity and lifetime value, the following set of issues have not been much addressed so far:

- maximizing customer lifetime value at the individual or segment level by explicitly linking the marketing actions of the company to the financial value generated in the long term, i.e. address the questions of whom to target, how, when, and why (economic motivation);
- estimating quantitatively the financial profile of customers, at an individual or segment level;
- using financial risk management techniques to select the optimal customer portfolio and allocate the limited marketing budget to maximize expected future value while minimizing the risk.

As to the first point, current and past literature has focused more on the measurement of average lifetime value (e.g. [25], [8], [40]), or on the identification of value drivers (e.g. [11], [34], [78]) which have an impact on the entire customer base. Neither lifetime value nor value driver heterogeneity across customers (or customer segments) is considered in most of the models.

Conceptual frameworks for customer asset management have been developed [7], [6]. However, these frameworks have a qualitative character and do not provide a quantitative model to maximize the customer value at an individual or segment level. As noted in [7], customer lifetime value should be viewed as an endogenous variable. This means that, in a model for the maximization of lifetime value, marketing actions influence customer lifetime value, which in turn has an impact on the type of marketing actions to select. In most of the models presented in the literature, however, customer lifetime value is seen as an exogenous variable, which does not depend on the marketing actions. Moreover, marketing resources are allocated over the entire customer base rather than across individuals or segments.
1.1. Problem description and relevance

For instance, models focusing on acquisition and retention (e.g. [34], [11]) assume a constant average lifetime value per customer and model the impact that retention and acquisition campaigns have on the number of new customers and old customers. As the number of customers increases the cumulative lifetime value, i.e. the customer equity, increases as well.

More complex stochastic models, such as the framework presented in [78], consider the average value generated by a customer by purchasing a given brand in a time step as exogenous. A logistic regression model is used to predict the probabilities of purchasing from a given brand if the last purchased brand is known. The coefficients of the regression are interpreted as the drivers of customer equity. Thus, the value drivers (e.g. quality, price, trust, etc.) can be identified and conceptually linked to marketing actions acting on these drivers, e.g. a loyalty campaign. By performing sensitivity analysis the impact of shifts in the drivers on the customer equity can be estimated. Sensitivity analysis is done by computing the partial derivatives of the purchase probabilities with respect to the drivers.

While this is a general methodology for the evaluation of the return on investment of a marketing initiative, from an operational point of view no explicit recommendation is given concerning which customer to target with which marketing action. While the lifetime value is computed at an individual level, the impact of value drivers is estimated on the entire customer base. Consequently marketing resources are not allocated at an individual or customer segment level. In fact, the same marketing action leading to the maximum return on investment, as measured by the increment in customer equity, is applied to each customer. Therefore heterogeneity in value drivers is not addressed. Furthermore, the estimation of the return on marketing is reliable only for small perturbations of the drivers around their nominal values, since first-order partial derivatives are used. In addition, the model does not address the issue of associating a value-driver shift with a marketing action. In fact, shifts in the drivers produced by particular marketing expenditures are an input of the model. A non-reliable estimate of these shifts would have a strong impact on the predicted return on marketing investment and, consequently, on the selection of the most appropriate marketing initiative.

Reliability issues lead to the second point: estimation of the financial profile. The literature focuses more on estimating the expected customer lifetime value. By aggregating this value the expected customer equity can be derived. From an investment-theory point of view, both expected value and the uncertainty about the future outcome of an investment should be considered when allocating resources [54], [13], [62]. In fact, of two investments with the same future expected outcome, the less risky is selected by the majority of decision makers. In general, a reduced volatility on the forecasted cash flow generates more value for the company since the cost of capital is reduced [15]. Therefore the value of an investment is deeply related to the uncertainty of the final outcome, i.e. the risk, of that investment.
a financial perspective, neglecting risk is a serious drawback.

The quantification of the uncertainty involved in forecasting lifetime value can therefore lead to a risk-sensitive allocation of marketing resources. In general, there are two sources of uncertainty. One is provided by the model used for the estimation of the lifetime value, which does not necessarily correspond to the “real” cash flow generation mechanisms. The second source of uncertainty is provided by the intrinsic stochastic nature of the customers’ behavior. According to these considerations, the lifetime value and the customer equity should be described by means of distributions rather than by exact numeric values. These distributions would take into consideration the underlying uncertainty (i.e. volatility) of the estimated quantities and allow for risk-sensitive decision-making.

In [78] the customer equity is computed by multiplying the average lifetime value per customer by the population size. To predict the customer equity of American Airlines, the average lifetime value computed using 93 customers is multiplied by a population of more than 43 millions of customers (market size). For such a large population, the sample size is probably too small to lead to a reliable estimate. For instance, if customers are very different, selecting another sample of 93 customers could potentially lead to a different average lifetime value estimation. The difference in average lifetime values would be amplified when computing the customer equity. A risk management approach would use the sample size and the market size to infer confidence intervals of the estimated quantity (i.e. the average lifetime value).

Taking into account the risks when allocating resources is a standard practice in finance, [54]. Investors are interested in maximizing the expected return on investment while minimizing the risk. Risk can be reduced through diversification by building a portfolio of different assets which are not much correlated\(^2\) [13], [27], [56].

The application of risk management techniques to customer assets leads to the third point: selecting the optimal customer portfolio for the marketing budget allocation. In finance, the concept of a portfolio is used as a means for reducing risk by diversification. In contrast, in the marketing literature the concept of customer portfolio does not involve risk, but indicates an aggregation of different customers, targets, or marketing initiatives. For example, in [11] the tradeoff between acquisition and retention investments is discussed and analytical tools are provided to allocate a marketing budget for acquisition and retention campaigns. The concept of portfolio refers in this case to the amount of money invested in the two campaign types. This amount is found by maximizing the expected customer equity: risk is not considered. In [41] customer portfolios are managed at a conceptual level. Three categories of customers are identified (acquaintances, friends, partners). In this case the concept of customer portfolios is used to describe an aggregation of customers. Consequently the lifetime

\(^2\)If the correlation between the return on investments of two assets is less than one, then combining the assets in a portfolio results in a lower standard deviation than investing all the available budget in one asset.
1.2 Contributions

In this thesis I develop advanced stochastic models addressing the three issues described in the previous section: a) maximization of customer lifetime value by linking marketing actions to the long-term financial value, b) estimation of the financial profile of customers, and c) use of risk management techniques to select the optimal customer portfolio. The general approach is illustrated in figure 1.1.

The dynamic interactions between the company and the customers are modeled using Markov Decision Processes [71]. The purpose of these models is to predict the life-cycle of the customers. Using dynamic programming [71] techniques it is possible to find the optimal sequence of marketing actions in time, i.e. the marketing policy, which maximizes the expected lifetime value of the customers at an individual level.

Traditionally, dynamic programming has been applied to optimize catalog mailing decisions (e.g. [10], [32]) where the decision variable is binary (mail or not mail). The issues of estimating robust Markov Decision Processes from customer relationship transactional data and of deriving the lifetime value distributions have not been addressed so far. To the best of my knowledge, with the exception of [82], all the models found in the literature assume a given state representation ad hoc. Most of the models are based on the recency, the frequency, and the monetary value (RFM) segmentation. While the RFM segmentation is very popular in the marketing practice [47], there is not a theoretical motivation that justi-
fies its use in modeling customer dynamics. As discussed in [82], the issue of estimating robust Markov Decision Processes is relevant because a non-reliable model can lead to a non-optimal policy, which in some cases could even perform worse than the historical, i.e. the current, marketing policy.

Using cross-validation [37] for model selection, a reliable Markov Decision Process is built which takes into account the impact of uncertainty in the parameters of the model on the performance measure (e.g. predicted lifetime value). In order to avoid making assumptions about the data generation process, e.g. linear utility function with additive noise following an extreme-value distribution, a structural dynamic process [77] modeling customer relationships is not estimated. As noted in [77], structural models suffer from parameter identification problems, i.e. there can be two different models that fit historical data equally well but yield very different predictions about the impact of a policy different than the historical policy which generated the data. Another drawback of parametric Markov Decision Processes is the high bias which can potentially lead to wrong marketing recommendations [82]. For these reasons I use a non-parametric approach to estimate the model. However, one of the drawbacks of non-parametric approaches is that such models cannot estimate the effect of marketing policies which have not been observed in the data [77]. To address this issue, a Bayesian approach when estimating transition probabilities has been adopted.

The model enables evaluation of the long-term effects of marketing actions at a customer segment level using Monte Carlo simulation. One can therefore consider lifetime value as an endogenous variable, which depends on the customer and on a sequence of marketing actions. This allows to address also heterogeneity in value drivers across customers: in fact, the optimal marketing policy is tailored to the individual customer. Simulation enables comparison of the impact that different marketing policies have on the customer equity.

The lifetime value distributions of customers are estimated using Monte Carlo simulation and bootstrap [26]. In this way both the uncertainty in the model parameters and the uncertainty in the customer behavior is captured. Then, using financial portfolio optimization techniques, one can investigate how to allocate a limited marketing budget to different customer segments so as to maximize the expected customer equity while minimizing the risk due to the uncertainty in the predictions.

In the last part of the thesis, I introduce another model based on mixture distributions to predict the lifetime value distributions of individual customers. Using utility theory [62], a customer portfolio maximizing expected utility is selected. In this case issues b) and c) are addressed. The lifetime value is not maximized using appropriate marketing actions. Rather, the customer portfolio is optimized by selecting those customers whose inclusion leads to an optimal value-risk tradeoff. I derive a closed form for the expected utility of each customer, which allows us to extend and generalize the traditional “rank and cut” practice used to build customer lists for the allocation of marketing resources [63]. This approach extends
1.3. Structure of the thesis

The thesis is organized as follows. In chapter 2, I review current analytical models for the estimation of lifetime value and customer equity and discuss the managerial implications that motivate the use of such models.

In chapter 3, I discuss how to model dynamic customer relationships using Markov Decision Processes. Once the model is estimated, I apply dynamic programming to efficiently allocate marketing resources and instruments in order to maximize the long-term value generated by customers in a given future time horizon. This methodology allows us both to predict and to maximize the future value generated by customers. Using bootstrap, one can estimate the lifetime value distributions. I illustrate the approach using data from a case study involving a major European airline, where I compare the performance of the historical, i.e. the current marketing policy, with the performance of the optimal marketing policy.

In chapter 4, I discuss how to build customer portfolios maximizing expected customer equity while minimizing the risk and satisfying the constraints on the available marketing budget. This chapter assumes that the lifetime value distribution of each segment (i.e. asset type) has been estimated using either the model described in chapter 3 or historical simulation. In the case of historical simulation, the marketing policy is not optimized since the history of customer transactions is used to estimate the distributions. The advantage of using Markov Decision Process models coupled with Monte Carlo simulation is that the lifetime
value distributions induced by the optimal marketing policy can be estimated. When evaluating the risk profile of customer assets there is not always an effective value-risk tradeoff. In contrast, in the case of financial assets, the value-risk tradeoff is induced by the laws of supply and demand governing the nearly efficient markets. I discuss the conditions under which a value-risk tradeoff is a relevant issue. For instance, uncertainty due to the model parameters can have a large impact on the value-risk tradeoff.

In chapter 5, I introduce an alternative model addressing the problems of predicting the future value distribution of individual customers and of allocating a limited marketing budget in order to build a customer portfolio that maximizes the expected future value while taking into account the uncertainty of that value. As marketing data is often sparse at the individual customer level [76], I propose to use Mixtures of Gaussians [86] in order to borrow strength from the data and estimate the individual probability densities. Once the future densities are known, utility theory is used for the allocation of marketing resources by taking into account the risk due to the intrinsic stochastic nature of the future predictions. Using data from a case study, I discuss several marketing applications, including the forecasting of future value, the identification of defectors, and the establishment of a customer portfolio. I show that the predictive performance of the mixture model is in line with that of state-of-the-art predictive algorithms and that a risk-sensitive resource allocation leads to customer portfolios with a lower downside risk.

Conclusions follow in chapter 6. Appendix A gives an overview of risk management and utility theory, the purpose is to introduce the concepts and analytic techniques used in chapters 4 and 5.
Chapter 2

Literature review

2.1 Introduction

The evaluation of customers’ value has been a relevant concern since the early 1940s when a few companies began to estimate the value of their customers. Later, at the end of the 1960s, companies started to use computer technology to predict the long-term value of customers. However, only recently with the capabilities of modern information technology infrastructures (e.g. databases, customer relationship management software, data mining, etc.) and the availability of customer data, has the concept of customer lifetime value been analyzed and defined more scientifically by focusing on the long-term relationship between the individual customer and the company [66].

The importance of the customer lifetime value concept is justified by the fact that every company has interest in retaining highly profitable customers, in reducing the number of customers which are not profitable, and in incrementing the value of the global customer base. Customers are not profitable if the costs generated to serve them exceed the revenues in the long term. Since the lifetime value approach considers the profitability of customers in the long term, decisions based on this methodology can lead to a high return on investment in the long term even if the short-term pay-off is negative (e.g. acquisitions of new customers). From a marketing perspective, investments in customers can take the form of loyalty campaigns aimed at retaining those customers with high lifetime value, acquisition campaigns aimed at converting potential high-value prospects into customers, and development campaigns designed to maximize the customers’ lifetime value by acting on its main drivers. When the entire customer base is considered, the individual lifetime value can be aggregated to build the customer equity of the company.

In this chapter I describe the state-of-the-art in analytic models for the prediction of customer lifetime value and the allocation of marketing resources. Particular emphasis is
Chapter 2. Literature review

given to the relationships between different models.

2.2 Customer lifetime value

Customer lifetime value (CLV) is defined as the sum of the discounted cash flows that a customer generates during his relationship with the company [8]. This definition can be expressed analytically with the following formula

$$\text{CLV}_j = \sum_{t=0}^{T} \frac{r_{j,t} - c_{j,t}}{(1 + i)^t}.$$  \hspace{1cm} (2.1)

Where \( j \) refers to the customer being evaluated, \( r_{j,t} \) and \( c_{j,t} \) are the revenue and the cost generated by customer \( j \) in time \( t \), \( i \) is the constant discount rate taking into account the time value of money, and \( T \) is the duration of the relationship with the company. The lifetime value approach considers customers as assets [12], [79], which generate cash flows over time and evaluates these assets using the discounted cash flow method which is a standard methodology for the financial evaluation of assets [15].

2.3 Customer equity

The concepts of customer equity (CE) and lifetime value (LTV) are related and often considered equivalent in the literature. While there is a common agreement on the definition of lifetime value, there are different definitions of customer equity. Some authors consider customer equity as the lifetime value less the acquisition cost [8], while others consider customer equity as the aggregated value of the entire customer base [40]. Analytically, customer equity is calculated with the following formula

$$\text{CE} = \sum_{i=1}^{N} \text{CLV}_i,$$

where \( \text{CLV}_i \) is the lifetime value of customer \( i \) and \( N \) is the total number of customers which includes the current customer base and the future customers.

The customer equity is therefore the present value generated by all the customers during their relationship with the company. Since most of a company’s cash flows are ultimately generated by its customers, the customer equity is a good proxy for the evaluation of the entire company. This holds especially in cases where the company’s assets are not tangible such as in the case of internet companies (e.g. Amazon, eBay, etc.). The correlation between customer equity and market value has been studied in [34] where customer equity is found to be less volatile than the market value. In the next section I analyze the managerial implications of the customer equity and the customer lifetime value.
2.4 Customer equity management

According to [8], customer equity management is a dynamic interactive marketing system that uses financial valuation techniques to optimize the acquisition, the retention, and the selling of additional products to the firm’s customers. Once analytical models for estimating customer equity and lifetime value are determined, drivers of customer equity can be identified and consequently exploited for targeted marketing actions. Researchers (e.g. [34]) have found, for instance, that a little improvement in the retention rate can have a large impact on the company’s customer equity. In general, the effectiveness of a marketing action can be measured by its return on investment according to the following formula

\[ \text{ROI}_{CE} = \frac{\Delta CE - I}{I}, \]

where \( \Delta CE \) is the incremental improvement in customer equity, due to the marketing action, and \( I \) is the present value of the investment associated with the marketing action.

When evaluating different marketing strategies, the one with the highest return on investment should be selected. An analogous formula can be applied for the calculation of the return on investment at the lifetime value level. In fact, the lifetime value of customers can be managed either at an individual level or at an aggregated level. In the first case, marketing decisions depend on the individual customer value (e.g. acquisition of certain customers rather than others). In the second case, marketing decisions are evaluated based on their impact on the entire customer base (i.e. customer equity). An example of customer equity based marketing decisions is the allocation of a marketing budget for acquisition and retention campaigns. In [11] the tradeoff between acquisition and retention spending, and criteria for the budget allocation are discussed. Since marketing budgets are always limited, there is the problem of deciding how much to allocate to different marketing actions. A company which invests too much in retention can limit the growth of its customer base, since all the efforts are directed towards keeping the current customers. The growth rate of the customer base affects, in turn, the customer equity and the market value of a company. On the other hand, a company investing only in the acquisition of new customers can lose very profitable customers to the competition and shrink its market share. Empirical studies [61] have demonstrated that the lifetime value is usually not constant among customers. In certain cases, 20% of the customers can generate up to 80% of the profits. Losing these customers would certainly have a negative impact on the company’s business.

The relationship between marketing actions and customer value is discussed from a conceptual but not quantitative perspective in [7]. In particular, the role of information technology to build a database enabling the computation of customer lifetime value is emphasized.
Chapter 2. Literature review

2.5 Customer lifetime value models

According to [25], lifetime value models can be classified into two categories: a) retention-based models, and b) migration-based models. The two models differ in the way in which they interpret the buyer-seller relationship. In the case of retention-based models, it is assumed that customers are committed to a given vendor because of high switching costs or other factors. In this context it is likely that if a customer decides to leave his supplier, he will never come back again. On the other hand, models based on the customer migration concept assume that customers have several suppliers and decide when to engage a relationship with a supplier. As a consequence, customers that leave in a given moment of time can return in the future. This kind of relationship has been defined as “polygamous loyalty” and is supported by low switching cost and by a high competitiveness among suppliers.

2.5.1 Retention-based models

The retention rate is the percentage of customers that remain loyal to the company each year. At an individual level, the retention rate can be interpreted as the probability of not defecting in a one-year period.

If the retention rate and the average contribution margin (i.e. revenue less variable costs) are constant, and the lifetime is fixed, then the average lifetime value per customer can be derived from formula (2.1):

$$\text{CLV} = (\bar{r} - \bar{c}) \cdot \sum_{t=0}^{T} \frac{\bar{p}}{(1 + i)^t}. \quad (2.2)$$

Where $\bar{r}$ is the average revenue per customer, $\bar{c}$ is the average cost per customer, $\bar{p}$ is the average constant retention rate, and $\bar{T}$ is the average lifetime. The above formula can be derived from (2.1) considering that cash flows do not depend on time and that at each step in time a given percentage of customers will defect (i.e. $1 - \bar{p}$). In practice, $(\bar{r} - \bar{c})$ can be considered as the average contribution margin obtained by dividing the total annual contribution margin by the number of customers active in the year, assuming that the time unit for the determination of lifetime value is in years. This approach considers customer lifetime value as an exogenous variable, not depending on marketing actions. Moreover, heterogeneity across customers is not addressed. An application of this model can be found in [33] and [34].

When profits change over time, then a function which models the profit trend can be estimated from the data [8]. Often revenues increase in time due to cross-selling and upselling, while costs to serve customers decrease. However, this is not a general pattern: in [72] the hypothesis that the margins of customers increase over time is rejected.
A generalization of the retention-based model is the survival-analysis-based model, in which the retention is interpreted as the probability of not defecting in time. In general, this probability is not constant but depends on the particular customer.

**Survival-analysis-based models**

The survival function is defined as the probability of being active in a given time frame. In a biological context, it refers to the probability of being alive after a given time. In the context of customer relationships, the survival function refers to the probability of not defecting during a given time interval.

The survival function can be calculated by knowing the hazard function \( h(t) \), which is defined as the probability of defecting at a given moment in time \( t \) given that the individual’s lifetime is no less than \( t \). If the hazard \( h_i(j) \) function of individual \( i \) is known, the survival function \( S_i(t) \) of individual \( i \) can be calculated as follows:

\[
S_i(t) = \prod_{j=0}^{t} (1 - h_i(j)).
\]

The hazard function can be computed using the Kaplan-Meier method such as in [23]:

\[
h(t) = \frac{d_t}{r_t},
\]

where \( d_t \) is the number of customers defecting at time \( t \), while \( r_t \) is the number of customers at defection risk at time \( t \). The hazard function can be interpreted as the defection rate at a given moment in time. In [23], artificial neural networks [37] are used to predict the survival function for each customer. In [81], the probability of being an active customer is computed using a negative binomial distribution (NBD/Pareto) model, which takes into account the fact that customer defection is not directly observed in a non-contractual setting. The probability of being active depends on the frequency, i.e. number of purchases, and the recency, i.e. time elapsed since last purchase, of the customer. The NBD/Pareto model is used in [72] and [74] to predict the lifetime of each customer.

Once the survival function and the average margin per customer in time have been estimated, the average customer lifetime value can be calculated as follows:

\[
\overline{CLV} = \sum_{t=0}^{T} S(t) \cdot m(t) \cdot d(t).
\]

Where \( S(t) \) is the average survival function, \( m(t) \) is the average margin at time step \( t \), \( d(t) \) is the discount factor to consider the value of time and the risk involved in future cash flow forecasts, and \( T \) is the average lifetime horizon.

Equation (2.3) can be seen as a generalization of the previously introduced equations. For instance, equation (2.2) can be obtained with the following assumptions:
• Constant retention rate: \( S(t) = p^r \).
• Constant average margin: \( m(t) = \bar{\tau} - \bar{c} \).
• Discount factor: \( d(t) = (1 + i)^{-t} \).
• Lifetime horizon: \( T = T \).

There are several methods for estimating the survival function and the average margin. In the case that these functions are continuous, equation (2.3) becomes:

\[
\text{CLV} = \int_0^T S(t) \cdot m(t) \cdot d(t) dt,
\]

where the discount rate is continuous, e.g. \( d(t) = e^{-\alpha t} \).

### 2.5.2 Migration-based models

Most of the customer migration-based models [25] use the recency of the last purchase to predict the probability of a subsequent purchase. Customers that are not active for a long time are therefore considered in the estimation of lifetime value. Migration models can be seen as a generalization of retention-based models. In fact, migration-based models assign a survival probability (i.e. a retention rate) to customers depending on their recency. The customer lifetime value is computed in two steps. Firstly, the number of customers who purchase in a given year is calculated. Secondly, the average lifetime value per customer is calculated.

Concerning the first step, the number of customers \( N_t \) who purchase in year \( t \) is calculated as follows:

\[
N_t = \sum_{j=1}^{t} [N_{t-j} \cdot p_j \cdot \prod_{k=1}^{j} (1 - p_{j-k})].
\]  

(2.4)

Where \( p_j \) is the probability of purchase given that the recency is \( j \) and \( p_0 = 0 \), \( N_0 \) is the number of initial customers. The above equation has a recursive form. If the initial number of customers is known (i.e. \( N_0 \)), it is possible to compute the expected number of customers in the following years (i.e. \( N_1, N_2, \) etc.). Once the future number of customers is known, the average lifetime value per customer can be calculated as follows:

\[
\text{CLV} = \frac{(\bar{\tau} - \bar{c})}{N_0} \cdot \sum_{i=0}^{T} \frac{N_t}{(1 + i)^i}.
\]  

(2.5)

Where \( i \) is the discount rate. This equation can be reformulated as a survival-analysis model. In fact, with the following substitutions, we obtain equation (2.3):
2.5. Customer lifetime value models

- \( S(t) = \frac{N_t}{N_0} \)
- \( m(t) = \tau - \bar{c} \)
- \( i(t) = (1 + i)^{-t} \).

Formula (2.4) calculates an average value, since the sum of all the lifetime values of the current customer base is divided by the total number of current customers. The total lifetime value, i.e. the customer equity without considering growth of the customer base, is simply obtained by multiplying the average lifetime value by the initial number of customers.

Markov chain-based models

Markov chain models [67] can be seen as a generalization of both retention and migration-based models. If we suppose that the probability of being active in the next time period depends not only on the recency but also on other features (e.g. frequency, monetary value, demographics, etc.), one can derive a more general and accurate model for lifetime value evaluation. A Markov chain is a sequence of random variables characterized by the fact that the value of the next variable depends only on the present and not the past, more formally:

\[
P(s_{t+1}|s_t, s_{t-1}, \cdots, s_0) = P(s_{t+1}|s_t).
\]

The values of the random variables are the states of the modeled system (e.g. customer states in this case). As an example of Markov chain modeling a customer, let us consider recency as the only state features and define the states as:

\[
\begin{cases}
S_1 & \text{if } \text{recency} = 1 \\
S_2 & \text{if } \text{recency} > 1.
\end{cases}
\] (2.6)

Next, the transaction probabilities from one state to another are defined as shown in figure 2.1. Customers who bought in the last time step are in state \( S_1 \). For these customers the probability of buying again in the next purchasing cycle is 0.3, while there is a 0.7 probability of defecting. Once a customer is in state \( S_2 \) (recency larger than 1) he will remain in that state.

Figure 2.1: A simple Markov chain specifying a customer retention-based model.
forever and never buy again. Note that the Markov chain in figure 2.1 specifies a retention-based model with a constant retention rate.

A simple migration model can be represented with the Markov chain depicted in figure 2.2. In this example, the states represent the recency since the last purchase. Customers with recency 5 are considered lost forever. For each recency value there is an associated probability of buying in the next time step. For example, a customer with recency 3 has a probability $p_3$ of buying in the next time step. The buying process is represented as a transaction leading to the state with recency value 1. Given the states and the transaction probabilities, if each state is associated with a cash flow, it is possible to forecast the sequence of expected cash flows in time. The expected lifetime value is computed as the sum of the discounted expected cash flows as follows:

$$CLV^T = \sum_{t=0}^{T} [(1 + d)^{-1} P]^t R.$$  

Where $P$ is the probability transition matrix whose element $p_{i,j}$ is the probability of moving from state $i$ to $j$. $R$ is the expected reward vector whose element $r_j$ is the expected cash flow associated with state $j$. Finally, $d$ is the discount rate and $T$ the lifetime.

For an infinite time horizon, if $\| (1 + d)^{-1} P \| < 1$, the expected lifetime value is [67]:

$$CLV^\infty = \lim_{T \to \infty} CLV^T = \{I - (1 + d)^{-1} P\}^{-1} R.$$  

The above quantity is in general easy to compute. The main issue with Markov chain models is the determination of the chain itself, i.e. the modeling of the states and the transaction probabilities. For this reason, simplifications such as retention-based models are more popular.

In [78] the customer states are defined as the brand of the last purchase and a multinomial logistic regression is used to predict the probabilities of switching to another brand if the current brand and some customer features are known. A similar approach is used in [21].

---

1The norm of a matrix $A$ is defined as $\|A\| = \max_{x} s.t. \|x\| = 1 \|Ax\|.$
where the states are defined based on the type of service purchased and the length of the relationships, then a probit model is used to predict the transition probabilities.

As discussed in [82], parametric models such as linear or logistic regression tend to have a larger bias than non-parametric models because they assume a given data generation process, e.g. linear function with additive random noise, which does not always correspond to the real data generation process. For instance, in the case of linear regression, a linear model for the conditional expected value of the output given the input features and an additive error following a Gaussian distribution with zero mean and constant variance are assumed. Using the maximum likelihood method or using statistical decision theory with a squared error loss function, the parameter vector is estimated by minimizing the residual sum of squares. Using the above assumptions, confidence intervals of the estimated parameters can be found. However, if the assumptions do not hold, the model can lead to a high bias and the inferences about the significance of the value drivers (i.e. regression coefficients) might be wrong.

In contrast, non-parametric models, e.g. nearest neighbors, are more flexible and can better adapt to the real unknown data generation function. However, these models tend to exhibit higher variance [37]. This variance can be estimated from the data using bootstrap [26] in contrast to the bias which cannot be estimated from the data [35].

2.6 Conclusions

Most of the lifetime value models described in the literature do not address heterogeneity in customer lifetime value. Moreover the issue of maximizing customer lifetime value at the individual or segment level is not explicitly addressed, since customer lifetime value is seen as an exogenous variable, i.e. not depending on the marketing action.

There are only a few examples that address the problem of estimating the customer lifetime value at an individual level. Individual lifetime value models use regression methods (e.g. linear regression, logistic regression, neural networks, etc.) to compute the expected cash flows and the survival (or migration) probabilities given individual features such as recency, frequency, monetary value, age, etc. Some examples of regression models can be found in [36], [23], [78], [21].

The Markov chain models are a generalization of both retention and migration-based models and can deal with individual lifetime value estimation, but the estimation of the underlying chain (i.e. states, transaction probabilities, rewards) has not been much addressed in the marketing literature. Current approaches use parametric regression techniques [78], [21] to predict the transition probabilities, but the states are chosen ad hoc and the expected rewards do not depend on the marketing actions. Parametric regression methods to predict
transition probabilities can lead to a high bias and consequently to wrong marketing recommendations as shown in [82]. The main issue with Markov chains is that these models measure lifetime value but do not explicitly maximize it by finding the best marketing actions at an individual customer level.

Another issue which has not been addressed thus far is the estimation of the uncertainty in the lifetime value prediction. Cash flows in time are not deterministic, and even if they were so, the lifetime value model is not guaranteed to predict them correctly. At present, there is no model which deals with risk by providing information about the volatility of the predicted lifetime values. Consequently, risk is not considered when allocating marketing resources.
Chapter 3

Modeling Customer Dynamics

3.1 Introduction

In the past years there has been an increasing interest in the allocation of marketing resources both in the marketing [34], [79], [50], and in the data-mining [75], [23], [29], communities. There is common agreement that marketing initiatives should be evaluated by measuring their impact on the customer lifetime value [40], i.e. the long-term value generated by a relationship with a customer.

In this chapter I discuss how to model the dynamic relationships between the customers and the company using Markov Decision Processes [71] and how to allocate efficiently the marketing budget by finding marketing actions that maximize the long-term value generated by the customers.

Markov Decision Processes model the dynamics of customer relationships by linking marketing actions with the long-term value (i.e. lifetime value) generated by customers during their life-cycle. Therefore lifetime value is seen as an endogenous variable which depends both on the customer and on the marketing actions. Markov Decision Processes allow the evaluation and optimization of the financial profile of customers using dynamic programming.

A Markov Decision Process (MDP) consists of a set of states, actions, transition probabilities, and reward functions. When an action is applied to a given state, the process moves stochastically to another state and generates a reward (e.g. cash flow). The probabilities of moving to a state given the origin state and the applied action, and the expected rewards generated in a given state when an action is applied, are part of the model specification. In this way a random sequence of states, actions, and rewards can be modeled and the expected cumulative reward associated with a given state under a specified policy, i.e. a mapping from state to actions, can be computed.
The use of dynamic programming techniques to maximize customer future long-term value and the concept of Markov Decision Processes itself originated from the catalog industry in the 1950s [38]. Next I give an overview of the literature.

In [10] customers are segmented according to their recency, i.e. elapsed time since last purchase, and frequency, i.e. total number of purchases in a given time interval. Then the Markov Decision Process and the optimal catalog mailing strategy satisfying the budget constraints are found. The decision variables are the number of customers to target in each segment and in each rental list, the number of catalogs to send, and the amount of money invested in inventory at each time step. A heuristic is used to solve the constrained multistage optimization problem, and the optimal sequence of mailing and inventory decisions maximizing the customer lifetime value is found. Using Monte Carlo simulations, the performance of the optimal mailing and reordering policy is evaluated. Moreover, the model can be used for risk analysis because simulations allow us to estimate the probability of running out of cash.

In [32] customers are segmented according to recency and frequency, and the transition probabilities between states are modeled according to an econometric utility based model. The optimal mailing policy is determined by optimizing a Markov game in which both the mailing decisions of the company and the ordering decisions of the customers mutually influence the utility function of the two players. As noted in [82], this approach depends on the specification of the utility function and assumes that the customers are aware of the optimal policy of the company in order to derive their own optimal policy. This assumption might not be realistic in practice, because, as revealed by the historical data, the company does not optimize its mailing policy and as a consequence the customers cannot infer the optimal mailing policy based on the available data. Finally, the model predicts only the purchasing probability but not the amount of the purchase.

Other examples of modeling customer relationship and maximizing lifetime value include [67] and [18], where customers are segmented according to their recency, frequency, and value. In [67] the Markov Chain modeling the relationship between the customers and the company is specified, then matrix algebra is used to forecast the future value of each state. Although Markov Decision Processes are not explicitly modeled, the improvement of the marketing policy using the policy improvement algorithm [71] is informally described. A marketing policy is a mapping from customer states to marketing actions. In [18] both infinite and finite time horizons are considered, and linear programming is used instead of dynamic programming in the infinite horizon case.

In [82] Markov Decision Processes are estimated directly from the data using a nonparametric approach that does not impose any functional form to the transition probabilities and the rewards. States are defined using a supervised clustering algorithm based on linear regression rather than the conventional RFM model. Dynamic programming is then used to
find the optimal mailing policy. Finally, a field test is implemented to show empirically, and not by simulation as in the above mentioned literature, the performance of the optimal and the historical mailing policies. As revealed by the test, the optimal policy does not always outperform the historical policy. The reason for this apparently unexpected finding is that some parameters of the model cannot be well estimated because of insufficient data, this leads to a biased estimate of the optimal state values when applying dynamic programming.

A different approach to solve the lifetime value maximization problem is reinforcement learning [84]. In [65] reinforcement learning with functional approximation is used to find the optimal targeting strategy, the advantage of using this method is that the Markov Decision Process does not need to be estimated from the data and that the feature space does not need to be discretized. However, the absence of a state-based model has the drawback of interpretability, if states can be associated with actions it is easier to manage the marketing investments by customer segment rather than at the individual customer level. Moreover, convergence to the optimal policy is not guaranteed in the case of reinforcement learning with functional approximation [84].

A main issue that has not been addressed in the literature is the estimation of robust Markov Decision Processes modeling the customer relationship and the long-term effects of marketing actions. To the best of my knowledge, with the exception of [82], all the models found in the literature assume a given state representation *ad hoc*, without providing any theoretical justification. Most of the models are based on the recency, the frequency, and the monetary value (RFM) segmentation. While the RFM segmentation is very popular in the marketing practice [47], there is no theoretical motivation that justifies its use in modeling customer dynamics. As discussed in [82], the issue of estimating robust Markov Decision Processes is relevant because a non-reliable model can lead to a non-optimal policy, which in some cases could even perform worse than the historical, i.e. the current, policy.

In this chapter I address the general issue of estimating a robust Markov Decision Process that best fits the data according to a performance criterion such as likelihood or long-term value prediction error. I use cross-validation to perform model selection and bootstrap to predict the future value. In this way the uncertainties both in the model parameters and in the predictive performance are captured.

As noted in chapter 1, I do not estimate a structural dynamic process [77] to model customer relationships because I do not want to make any assumption about a given data generation process, e.g. linear utility function with additive noise following an extreme-value distribution. As noted in [77], structural models suffer from parameter identification problems, i.e. there can be two different models that fit historical data equally well but yield very different predictions about the impact of a policy different than the historical policy which generated the data. Another drawback of parametric Markov Decision Processes is the high bias, which can potentially lead to wrong marketing recommendations [82]. For
these reasons I use a non-parametric approach to estimate the model. However, one of the drawbacks of non-parametric approaches is that such models cannot estimate the effect of marketing policies which have not been observed in the data [77]. To address this issue, I adopt a Bayesian approach when estimating transition probabilities.

The remainder of the chapter is organized as follows. In section 3.2, I describe, by means of an example, how the dynamic interactions between the company and its customers can be modeled using Markov Decision Processes. In section 3.3 I briefly review Markov Decision Processes and dynamic programming. In section 3.4 I describe how to estimate robust Markov Decision Processes from the customer transactional data. In section 3.5 I introduce the methodology for model selection, based on cross-validation. In section 3.6 I discuss how to evaluate marketing policies and how to estimate the financial profile of customers. In section 3.7 I discuss how to optimize the marketing policy and consequently the customer equity and the return on marketing investment. In section 3.8 I describe a case study in which the methodology to estimate a Markov Decision Process modeling the customer dynamics is applied. I use the estimated model to predict and maximize the future long-term value. The conclusions follow in section 3.9.

3.2 Modeling customer relationships

I model the customer behavior in time taking into account the marketing actions executed by the company. Figure 3.1 depicts an example of a Markov Decision Process modeling the interactions between the customers and the marketing actions.

Figure 3.1: Example of modeling customer dynamics with Markov Decision Processes. The customers are segmented in 3 states. The effects of marketing actions are modeled by the transition probabilities and the expected rewards.
Customers are segmented in states, i.e. $S_1$, $S_2$, $S_3$. State $S_1$ gathers those customers that bought only once or that did not buy since a long time. Customers in state $S_2$ have a buying frequency larger than one and a not too large recency. Finally, customers in state $S_3$ have a high consumption pattern, these customers are very loyal and buy frequently.

The Markov Decision Process is fully specified by associating to each state the possible marketing actions, the transition probabilities and the expected rewards. The reward is defined as the net cash flow obtained by applying an action in a given state.

According to model defined in figure 3.1, if a special offer is sent to a customer in state $S_1$ there is a 0.7 probability that the customer will accept the offer. By accepting the offer the customer moves to state $S_2$ and generates an expected reward of -27, due to the cost of the promotion. If customers in state $S_1$ are not targeted, there is a lower probability of moving to state $S_2$ but the company will not allocate marketing budget to promote a special offer.

At each time step, i.e. month, the company can decide to target the customers with the marketing actions allowed in each state. The action “do nothing” is explicitly modeled, since it represents a decision which has a different effect on the relationship than sending a special offer.

In order to efficiently allocate the marketing resources, the actions which maximize the expected long-term value, i.e. lifetime value, generated by the customers should be selected. Once all the states, the transition probabilities, and the expected rewards are known, it is possible to find the marketing policy which maximizes the expected long-term value generated by the relationship with the customers.

For instance, assuming that customers in state $S_3$ are very profitable (their immediate expected profit is $0.8 \times 50 = 40$), it can be profitable to send a special offer in state $S_1$ and a club membership offer in state $S_2$ even if the immediate expected rewards are negative. The expected reward of sending a special offer to customers in state $S_1$ is $-27 \times 0.7 - 2 \times 0.3 = -20.4$ while the expected reward of sending a club membership offer to customers in state $S_2$ is $-100 \times 0.7 - 5 \times 0.3 = -71.5$. Therefore, a short-sighted marketing policy would not send any offer to customers in states $S_1$ and $S_2$, however such a policy would not take into consideration the long-term value generated by upgrading customers to state $S_3$. As shown in section 3.3.1 such a policy is not always optimal. Once a marketing policy is fixed, the future customer dynamics can be simulated for a given time horizon and the distribution of the future values can be estimated.

### 3.3 Markov Decision Processes

A Markov Decision Process (MDP) can be defined as follows [71].
Definition 3.3.1 (Markov Decision Process). A finite state Markov Decision Process (MDP) is a discrete time dynamic process defined by

- a set of decision epochs $T \subset \mathbb{N}$,
- a finite set of states $S$,
- a finite set of actions $A$,
- a transition probability $p_t(s'|s, a)$ modeling the probability of moving from state $s \in S$ to state $s' \in S$ if action $a \in A$ is applied at decision epoch $t \in T$,
- a reward function $r_t(s, a)^1$ modeling the expected reward obtained in state $s \in S$ if action $a \in A$ is applied at decision epoch $t \in T$.

If the transition probabilities and the rewards do not depend on the decision epoch, the process is said to be stationary. I consider stationary Markov Decision Processes when modeling the dynamics of customer behavior.

### 3.3.1 Dynamic programming

Markov Decision Process theory is based on using backward induction (dynamic programming) to recursively evaluate expected rewards [71]. A deterministic\(^2\) policy $\pi$ defines, for each decision epoch $t \in T$, a decision rule $d_t$ mapping states to actions, i.e.

$$\pi = \{d_1, \cdots, d_t, \cdots\},$$

with

$$d_t(s_t) \rightarrow a_t$$

where $s_t$ indicates state $s$ at decision epoch $t$, the same notation is used for the action.

Once the policy is fixed, the Markov Decision Process becomes a Markov Chain, i.e. a stochastic sequence of states and rewards, where the transition probabilities to the next state $s'$ and the reward do not depend on the action $a$ but only on the current state $s$, i.e.

$$p_t(s'|s) = p_t(s'|s, d_t(s)),$$

$$r_t(s) = r_t(s, d_t(s)).$$ (3.1)

---

\(^1\)For the sake of clarity, in figure 3.1 I use the expected reward $r_t(s, a, s')$, which depends as well on the destination state $s'$. The relationship with the expected reward defined here is $r_t(s, a) = \sum_{s' \in S} p_t(s'|s, a) r_t(s, a, s')$. For optimization and simulation the knowledge of $r_t(s, a)$ is sufficient [71].

\(^2\)Deterministic policies are a special case of stochastic policies, which associate to each state a probability distribution on the actions.
A policy can be selected according to an objective function. \( \Pi \) is defined as the set of all the possible policies. The policy \( \pi^*_s \), maximizing the expected value of the sum of the rewards, generated in a finite horizon of length \( T = |T| \), given the initial state \( s_1 \), is defined as

\[
\pi^*_s = \arg \max_{\pi \in \Pi} E_{s_1}^\pi \sum_{t=1}^{T-1} r_t(s_t, a_t) + r_T(s_T).
\]

(3.2)

\( r_t, s_t, \) and \( a_t \) are, respectively, the expected reward, the state, and the action executed at time step \( t \). \( r_T(s_T) \) is the terminal reward obtained at the last epoch \( T \) which depends only on the state \( s_T \).

The expectation is conditioned on the initial state \( s_1 \). Note that for a given policy, the sequence of states and rewards is random.

A similar objective function is given by the expected value of the sum of the discounted rewards generated during an infinite horizon, the optimal policy is then given by

\[
\pi^*_s = \arg \max_{\pi \in \Pi} E_{s_1}^\pi \left[ \sum_{t=1}^{\infty} \gamma^{t-1} r_t(s_t, a_t) \right],
\]

(3.3)

where \( 0 \leq \gamma < 1 \) is the discount factor which can be interpreted as a weight giving less importance to future outcomes, or as the present value of future cash flows. If the expected reward is bounded, then the sum converges to a finite value since \( |\gamma| < 1 \).

Generally, assuming that the expected rewards are bounded, the following results hold for Markov Decision Processes and the policies defined in (3.2) and (3.3) (the proofs can be found in [71]): a) the optimal policy exists and does not depend on the initial state, b) there is a deterministic optimal policy which does not depend on the history but only on the current state, and c) the optimal policy is in general time dependent in the finite horizon case and stationary in the infinite horizon case.

Since I consider a finite time horizon when modeling customer relationships, I focus on the case of optimal policies satisfying the objective function (3.2).

The expected value of state \( s \) at time step \( t \), given a policy \( \pi \), in the case of a finite horizon of length \( T \) is defined as

\[
\nu^\pi_T(s_t) = E_{s_t}^\pi \left[ \sum_{t=i}^{T-1} r_t(s_t, a_t) + r_T(s_T) \right],
\]

if the policy is deterministic, the expectation is computed over all the possible sequences of states, since the actions are deterministically associated with the states. In this case the above
equation can be developed as follows\(^3\)

\[

\nu_T^\pi(s_i) = \mathbb{E}^\pi_s\left[ \sum_{t=i}^{T-1} r_t(s_t, a_t) + r_T(s_T) \right] \\
= \mathbb{E}^\pi_s\left[ r_i(s_i, a_i) + \sum_{i+1}^{T-1} r_t(s_t, a_t) + r_T(s_T) \right] \\
= r_i(s_i, a_i) + \mathbb{E}^\pi_s\left[ \sum_{t=i+1}^{T-1} r_t(s_t, a_t) + r_T(s_T) \right] (3.4) \\
= r_i(s_i, a_i) + \sum_{s_{i+1} \in S} p_i(s_{i+1}|s_i, a_i) \mathbb{E}^\pi_{s_{i+1}}\left[ \sum_{t=i+1}^{T-1} r_t(s_t, a_t) + r_T(s_T) \right].
\]

Therefore the value in each state can be computed by \textit{backward induction} as follows

\[

\nu_T^\pi(s_T) = r_T(s_T), \forall s \in S \\
\nu_T^\pi(s_i) = r_i(s_i, a_i) + \sum_{s_{i+1} \in S} p_i(s_{i+1}|s_i, a_i) \nu_T^\pi(s_{i+1}), \forall s \in S, \forall i \in [1, T-1]. (3.5)
\]

From equation (3.5) one can note that a multistage problem is decomposed into a sequence of single-stage problems. Backward induction is the essence of \textit{dynamic programming} \cite{bellman1957}. The same principle can be applied in order to find the optimal policy \(\pi^*\) for a finite horizon MDP. The optimal policy starting from state \(s\) is defined as

\[

\pi^*_s = \arg \max_{\pi \in \Pi} \nu_T^\pi(s),
\]

since the optimal policy does not depend on the initial state \cite{bellman1957}, \(\pi^*_s = \pi^*\). Using backward induction the optimal value and the optimal policy can be computed by the recursive algorithm 1, note that the decision rules \(d_t\) do not depend on a particular initial state of the Markov Decision Process. It is possible to prove by induction \cite{bellman1957} that the algorithm converges to the optimal policy maximizing the objective function (3.2) for each choice of initial state \(s_1\).

### 3.3.2 Numeric example

By applying algorithm 1 to the MDP defined in figure 3.1, assuming that the time steps are expressed in months, one can find the optimal policy maximizing the value that customers

\(^3\)A similar relationship can be developed in the case of stochastic policies.
Algorithm 1 Backward induction

Require: MDP model
Ensure: optimal policy $\pi^*$ and optimal state value $\nu^*_{T}(s)

1: set $t = T$ and $\nu^*_{T}(s_T) = r_T(s_T), \forall s \in S$

2: for $t = T - 1$ to 1 do

3: $\nu^*_{T}(s_t) = \max_{a_t \in A} (r_t(s_t, a_t) + \sum_{s_{t+1} \in S} p_t(s_{t+1}|s_t, a_t) \nu^*_{T+1}(s_{t+1}))$, $\forall s \in S$

4: $d_t(s_t) = \arg \max_{a_t \in A} (r_t(s_t, a_t) + \sum_{s_{t+1} \in S} p_t(s_{t+1}|s_t, a_t) \nu^*_{T+1}(s_{t+1}))$, $\forall s \in S$

5: end for

6: $\pi^* = \{d_1, \cdots d_t, \cdots d_{T-1}\}$

generate during one year. I set $T = 13$ and the terminal reward $\nu^*_{T}(s_{13}) = r_{13}(s_{13}) = 0$, $\forall s \in S$. Using algorithm 1 for $t = 12$ and state $S1$ I obtain

$$\nu^*_{13}(S1_{12}) = \max_{a_{12} \in A} (r_{12}(S1_{12}, a_{12}) + \sum_{s_{13} \in S} p_{12}(s_{13}|S1_{12}, a_{12}) \nu^*_{13}(s_{13}))$$

$$\begin{align*}
&= 2 \quad (d_{12}(S1_{12}) = \text{do nothing}).
\end{align*}$$

I indicate in parenthesis the optimal action maximizing the value, i.e. “do nothing”. For state $S2$ I obtain

$$\nu^*_{13}(S2_{12}) = \max_{a_{12} \in A} (r_{12}(S2_{12}, a_{12}) + \sum_{s_{13} \in S} p_{12}(s_{13}|S2_{12}, a_{12}) \nu^*_{13}(s_{13}))$$

$$\begin{align*}
&= 12 \quad (d_{12}(S2_{12}) = \text{do nothing}).
\end{align*}$$

Finally, for state $S3$ I obtain

$$\nu^*_{13}(S3_{12}) = \max_{a_{12} \in A} (r_{12}(S3_{12}, a_{12}) + \sum_{s_{13} \in S} p_{12}(s_{13}|S3_{12}, a_{12}) \nu^*_{13}(s_{13}))$$

$$\begin{align*}
&= 40 \quad (d_{12}(S3_{12}) = \text{do nothing}).
\end{align*}$$

By iterating this procedure at each time step and for all the states, I obtain the optimal policy described in table 3.1.

<table>
<thead>
<tr>
<th>State</th>
<th>$d_t, t \in [1, 3]$</th>
<th>$d_t, t \in [4, 6]$</th>
<th>$d_t, t \in [7, 12]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>special offer</td>
<td>do nothing</td>
<td>do nothing</td>
</tr>
<tr>
<td>S2</td>
<td>club membership</td>
<td>club membership</td>
<td>do nothing</td>
</tr>
<tr>
<td>S3</td>
<td>do nothing</td>
<td>do nothing</td>
<td>do nothing</td>
</tr>
</tbody>
</table>

Table 3.1: Optimal policy for the MDP defined in figure 3.1, assuming 13 decision epochs and zero terminal rewards.

Note that for a short-time investment horizon, $t \in [7, 12]$, the optimal policy is to not send any offer to any customer. Sending a club membership offer to customers in state $S2$
pays off only after at least 6 months, while sending a special offer to customers in state $S1$ pays off only after at least 9 months. The optimal values generated in one year starting from the different states are $\nu_{13}^*(S_{11}) = 59.74$, $\nu_{13}^*(S_{21}) = 100.48$, and $\nu_{13}^*(S_{31}) = 215.2$, this values can be interpreted as the optimal lifetime value generated using the optimal marketing policy.

Figure 3.2 compares the performance of the optimal policy and the policy which never targets any customer. As performance measure I use the average value per state. If customers are not distributed equally among states, then the effective difference in value might be even larger. The value at month 1 is equal to the cumulative value generated in the next 12 months, the value at month 2 is equal to the cumulative value generated in the remaining 11 months (starting from month 2 until month 12) and so on.

The long-term values under the “always do nothing” policy $\pi$ are $\nu_{13}^*(S_{11}) = 44$, $\nu_{13}^*(S_{21}) = 64$, and $\nu_{13}^*(S_{31}) = 215.2$. In average, the optimal policy $\pi^*$ boosts long-term value by 17.29% compared to $\pi$.

Note that for time steps larger or equal than 7 the two policies are the same (table 3.1) and generate therefore the same average value per state (figure 3.2).
3.4 Estimation of the MDP

I assume that customers are segmented into different states and that customer transactional data is available. For each customer, the transactional data consists of a sequence of events. Each event is defined by a triple composed of a state $s$, an action $a$, and a reward $r$. The next event is defined by the triple $s', a', r'$ where $s'$ is the state resulting from applying $a$ to $s$ and so on. Each customer has an associated sequence of events defined as episode. Figure 3.3 defines the structure of an episode for a given customer.

![Figure 3.3: Example of an episode for a given customer. The initial state is B, after getting a mail, the generated reward is -2 and the next state is B. Then an email causes the customer to move to state A and a reward of 30 is generated, and so on.](image)

In order to completely specify the Markov Decision Process one needs to define the state space, the action space, the transition probabilities and the expected rewards. I use a non-parametric approach to obtain non-biased estimates of the transition probabilities and the expected rewards.

Given transactional data $D$, the state and action spaces are obtained by considering respectively all the states and all the actions that appear in $D$. In order to estimate the transition probabilities I use the maximum likelihood estimator:

$$p(s'|s, a) = \frac{\#(s'|s, a)}{\#(s, a)} \quad (3.6)$$

where $\#(s'|s, a)$ is the total number of transitions from $s$ to $s'$ if action $a$ is applied and $\#(s, a)$ is the total number of actions $a$ applied to state $s$. If the quantity $\#(s, a)$ is zero, then the above equation is not defined. Moreover, if the quantity $\#(s'|s, a) = 0$ then $p(s'|s, a) = 0$. As the transition probabilities are estimated from a limited sample of data, one can assume that the absence of particular transitions does not necessarily imply that the real probabilities are undefined or null. To address these two issues I adopt a Bayesian approach incorporating...
the prior transition probability $\hat{p}_{s'|s,a}$ into equation (3.6). This leads to the following estimator [59]:

$$p(s'|s, a) = \frac{\#(s'|s, a) + m_1 \hat{p}_{s'|s,a}}{\#(s, a) + m_1}. \quad (3.7)$$

It is easy to verify that if $\#(s, a) = 0$ then $\#(s'|s, a) = 0$ and $p(s'|s, a) = \hat{p}_{s'|s,a}$. Therefore, if no observations are available I use the prior probability $\hat{p}_{s'|s,a}$. The quantity $m_1$ can be interpreted as the number of instances following the prior probability that are injected into the data set $D$, $m_1$ acts therefore as a weight defining the relative importance of the prior probability with respect to the probability estimated from the data.

There are two possibilities to model the prior transition probability $\hat{p}_{s'|s,a}$: a) adopt a \textit{state-driven} approach, emphasizing the role of the origin state, or b) adopt an \textit{action-driven} approach, emphasizing the role of the action.

### 3.4.1 State-driven approach

In the first case, the prior is modeled as follows:

$$\hat{p}_{s'|s,a} = p(s'|s) = \frac{\#(s'|s)}{\#(s)} + \frac{m_2 \hat{p}_{s'|s}}{\#(s)},$$

where $\#(s)$ is the number of times state $s$ appears in the data set $D$ and $\#(s'|s)$ is the number of times a transition from state $s$ to state $s'$ is observed. Finally, the nested prior $\hat{p}_{s'|s}$ is estimated as follows:

$$\hat{p}_{s'|s} = p(s') = \frac{\#(s') + m_3 \hat{p}}{\sum_{s \in S} \#(s) + m_3}.$$

If state probabilities are uniformly distributed, then the prior $\hat{p}$ becomes

$$\hat{p} = \frac{1}{|S|},$$

where $|S|$ is the cardinality, i.e. the number of different states, of the set $S$. If $m_3 = |S|$, the Laplace estimator [88] is obtained:

$$\hat{p}_{s'|s} = p(s') = \frac{\#(s') + 1}{\sum_{s \in S} \#(s) + |S|}.$$

I use this estimator to model the prior $\hat{p}_{s'|s}$. I use the Laplace estimator to avoid the case $p(s') = 0$, which occurs when $\#(s') = 0$. When the model is estimated from the available data (learning phase) then $\#(s') > 0$, as each state is observed at least once in order to appear in the set of states $S$. However, if the model is tested using an independent test set (not used to train the model), a new state might appear.
3.4. Estimation of the MDP

3.4.2 Action-driven approach

In the second case, the prior is modeled as follows:

\[ \hat{p}_{s'|a} = p(s'|a) = \frac{\#(s'|a) + m_2 \hat{p}_{s'|a}}{\#(a) + m_2}, \]

where \( \#(a) \) is the number of times action \( a \) appears in the data set \( D \) and \( \#(s'|a) \) is the number of times a transition to state \( s' \) is due to the execution of action \( a \). The nested prior \( \hat{p}_{s'|a} \) is estimated as in the case of the state-driven approach using the Laplace estimator

\[ \hat{p}_{s'|a} = p(s') = \frac{\#(s')}{\sum_{s \in S} \#(s) + |S|}. \]

3.4.3 Reward

The expected reward \( r(s, a) \) if action \( a \) is applied to state \( s \) can be estimated as follows:

\[ r(s, a) = \frac{\sum_{(s,a) \in D} r(s,a)}{\#(s,a)}, \quad (3.8) \]

where \( r(s,a) \) is the reward observed in the data when action \( a \) is applied to state \( s \). If the quantity \( \#(s,a) \) is zero, because action \( a \) has never been applied to state \( s \), the expected reward can be estimated considering either a state-driven approach or an action-driven approach. The state-driven estimate is

\[ r(s, a) = r(s) = \frac{\sum_{s \in D} r(s,a)}{\#(s)}, \]

whereas the action-driven estimate is

\[ r(s, a) = r(a) = \frac{\sum_{a \in D} r(s,a)}{\#(a)}. \]

As the expected rewards do not include any information about the uncertainty and the non-deterministic nature of the rewards, I also estimate the empirical cumulative density function \( r(\cdot|(s, a)) \) if action \( a \) is applied in state \( s \). In this way, the likelihood of a sequence of states, actions, and rewards can be computed. Moreover the customer dynamics can be simulated.

Given a sample of \( n \) data points \( x_1, \ldots, x_n \) extracted from the cumulative density function (cdf) \( F(x) \), the empirical cumulative density function (ecdf) \( \hat{F}_{(n)}(x) \) is defined as [60]

\[ \hat{F}_{(n)}(x) = \frac{1}{n} \sum_{i=1}^{n} I_{(-\infty,x]}(x_i), \quad (3.9) \]
where $\mathbb{I}_{(-\infty,x]}(x_i)$ is equal to 1 if $x_i \in (-\infty,x]$ and equal to 0 otherwise. It is possible to prove that $\hat{F}_n(x)$ converges in probability to the real distribution $F(x)$ if $n \to \infty$. For large data sets, it might be very memory intensive to store $\hat{F}_n(x)$ explicitly. Therefore, to reduce the memory requirements I linearize the empirical cumulative density function piecewise as follows. I divide the number of observations into $k$ bins that all contain the same number of observations. Each bin is defined by its extremes $[x_{k-1}, x_k]$; because each bin has the same number of data observations, all bins have the same probability and induce, by the cumulative density function, a partition on the interval $[0,1]$ into $k$ bins of equal length with extremes $[\hat{F}_n(x_{k-1}), \hat{F}_n(x_k)]$. A linear function is associated to the $k$-th bin that is defined as:

$$y = \frac{\hat{F}_n(x_{k-1}) - \hat{F}_n(x_k)}{x_{k-1} - x_k} \cdot (x - x_k) + \hat{F}_n(x_k).$$

(3.10)

The number of bins $k$, which define the discretization level, is a parameter of the algorithm. Figure 3.4 compares the cumulative density function of a normal distribution with mean 10 and variance 4 to the piecewise linear approximation of the empirical cumulative density function defined in equation (3.9). The linear functions are defined according to equation (3.10), and the sample used to estimate the empirical cumulative density function consists of 2000 data points. As shown in figure 3.4, a piecewise linear model of 100 bins approximates very well the real distribution. Note that if $k = 100$ one needs to store just 200 data points $(x_k, \hat{F}_n(x_k))$ to memorize the piecewise linear model rather than storing 2000 values to explicitly represent the empirical cumulative density function.

For each state-action couple $(s,a)$ I consider the set $D_{(s,a)}$ of rewards generated in state $s$ when action $a$ is applied:

$$D_{(s,a)} = \{r : (r,s,a) \in D\}.$$

Once this set of rewards has been filtered from the data, I use it to estimate the empirical cumulative density function $r(\cdot | (s,a))$ using equations (3.9) and (3.10) in order to build the piecewise linear model. Again, as in the case of the estimation of the expected value, if there are no observations for a given state-action couple, I consider either a state-driven approach or an action-driven approach by modeling the distribution of $r(\cdot | s)$ and $r(\cdot | a)$ using the appropriate learning sets $D_{(s)} = \{r : (r,s,a) \in D\}$, and $D_{(a)} = \{r : (r,s,a) \in D\}$, respectively.

### 3.4.4 Estimation of the historical policy

The historical policy is defined as the policy used by the company when targeting customers. The knowledge of the historical policy allows us to simulate the customer dynamics. In fact, once the MDP and the historical policy are estimated from the available data, it is possible
3.4. Estimation of the MDP

Figure 3.4: Piecewise linear empirical cumulative density function (ecdf) compared with the real cumulative density ($\mathcal{N}(10, 2^2)$) function for two values of $k$.

to obtain the Markov Chain (using equations (3.1)) which allows us to model how customers evolve in a given time horizon.

From the available transactional data a stochastic policy is learned, assuming that it is stationary\textsuperscript{4}, and estimate the probability of executing action $a$ in state $s$ as follows:

$$
\pi(a|s) = \frac{\#(a|s) + m\hat{\pi}_a|s}{\#(s) + m},
$$

where $\#(a|s)$ are the number of events (i.e. transactions) with state $s$ and action $a$ and $\#(s)$ are the number of events with state $s$. The quantity $\hat{\pi}_a|s$ is the prior probability of executing action $a$ in state $s$. I define the prior probability using the Laplace estimator as follows

$$
\hat{\pi}_a|s = p(a) = \frac{\#(a) + 1}{\sum_{a \in A} \#(a) + |A|},
$$

where $\#(a)$ is the total number of actions of type $a$ in all the events and $|A|$ is the number of available actions.

\textsuperscript{4}This assumption is realistic if the company is not using any multistage decision model to target the customer base.
I consider the number of events in all the decision epochs (i.e. time steps), since I do not condition on a particular epoch. However, I could estimate a time-dependent policy by computing the above probabilities for each different time step.

\section*{3.5 Model selection}

A critical issue in predictive modeling is the assessment of the model’s performance. The ideal procedure to obtain a realistic estimate of the performance is to use a large independent test set which has not been used to learn the MDP. However, in order to take the most advantage of the available data other procedures such as cross-validation can be used \cite{37}. Once a methodology to assess the performance of a model has been defined, it can be used to select the model with the best performance.

I use cross-validation for model selection and consider the following two categories of performance measures: likelihood of the model and prediction errors. The likelihood evaluates the entire sequence of state, actions, and rewards. The prediction error evaluates the accuracy in predicting the long-term value generated by customers.

\subsection*{3.5.1 Log-likelihood}

The likelihood $\mathcal{L}$ of an episode is defined as the probability of observing the entire sequence of states, actions, and rewards defining the episode. The episode of customer $i$ is defined as

$$e_i = [(s_1, a_1, r_1), \ldots, (s_{T-1}, a_{T-1}, r_{T-1}), (s_T, r_T)].$$

Since in the final state no more actions are allowed, I set the terminal reward to zero and consider only the controllable events generated from the decision epoch 1 to the decision epoch $T - 1$.

I assess the predictive performance of the Markov Chain, modeling the dynamic behavior of the customers, which has been obtained by applying the historical policy to the estimated Markov Decision Process. I compute the likelihood until time step $T - 1$, conditioned on the initial state. In this way I do not take into consideration how well the model predicts the prior state distributions. Instead I focus on the controllable path starting from $s_1$ and ending at $s_{T-1}$. For numerical accuracy I compute the logarithm of the likelihood, i.e. the
3.5. Model selection

log-likelihood. Given a policy $\pi_t$, the conditional log-likelihood of the episode $e_i$ is equal to

$$
\log(\mathcal{L}(e_i)) = \log P[(s_1, a_1, r_1), (s_2, a_2, r_2), \cdots, (s_{T-1}, a_{T-1}, r_{T-1})|s_1] \\
= \log(\pi_1(s_1)r_1|s_1, a_1) \prod_{t=2}^{T-1} p(s_t|s_{t-1}, a_{t-1}) \pi_t(a_t|s_t)r(r_t|s_t, a_t) \\
= \log(\pi_1(s_1)) + \log(r(r_1|s_1, a_1)) + \sum_{t=2}^{T-1} \log(p(s_t|s_{t-1}, a_{t-1})) \\
+ \sum_{t=2}^{T-1} \log(\pi_t(a_t|s_t)) + \sum_{t=2}^{T-1} \log(r(r_t|s_t, a_t)).
$$

(3.11)

In formula 3.11 I consider a stationary Markov Decision Process with a time dependant policy. If the policy is stationary, e.g. historical policy, then $\pi_t = \pi$. The probability of executing action $a_t$ in state $s_t$ at time step $t$ is $\pi_t(a_t|s_t)$, the probability of moving to state $s_t$ when action $a_{t-1}$ is applied to state $s_{t-1}$ at time step $t$ is $p(s_t|s_{t-1}, a_{t-1})$. Finally, $r(r_t|s_t, a_t)$ is the value of the probability density function of the reward at point $r_t$ given state $s_t$ and action $a_t$ at time step $t$.

I derive the probability density function $\hat{f}$ from the estimated cumulative density function $\hat{F}$ by numerical derivation as follows

$$
\hat{f}(x) = \frac{\hat{F}(x + \epsilon) - \hat{F}(x - \epsilon)}{2\epsilon},
$$

where $\epsilon > 0$ is a small constant, in practice I use the following heuristic for determining $\epsilon$:

$$
\epsilon = \min \frac{|x|}{10} + 0.001,
$$
in this way $\epsilon$ is positive and not too small compared with the order of magnitude of the data.

Assuming the different episodes are independent, the log-likelihood of a set of episodes $D$ is

$$
\log(\mathcal{L}(D)) = \sum_{e_i \in D} \log(\mathcal{L}(e_i)).
$$

According to the maximum likelihood principle, the model with the highest likelihood should be selected.

3.5.2 Numeric prediction errors

The mean absolute error (MAE) is defined as

$$
MAE = \frac{\sum_{i=1}^{n} |y_i - \hat{y}_i|}{n},
$$
where \( y_i \) is the observed value of the \( i \)-th instance and \( \hat{y}_i \) is the prediction. The mean squared error (MSE) is defined as

\[
MSE = \frac{1}{n} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2.
\]

The mean squared error penalized data sets with a high variance more than the mean absolute error.

### 3.5.3 Methodology for the model selection

Several parameters influence the estimation of a Markov Decision Process modeling customer relationships, such as the segmentation used to define the customer states, the state-driven or action-driven approach to estimate the transition probabilities and the rewards, the length of the time horizon in the training data, etc. The definition of the state space will probably have the highest impact on the performance of the model because the transition probabilities, the rewards, and the historical policy are estimated based on the states encountered in the training data.

Assuming there is a finite list of possible models \( M_1, \ldots, M_n \), corresponding to different choices of the parameters, I use the algorithm 2 in order to perform model selection.

---

**Algorithm 2** Model selection using cross-validation.

**Require:** Set of \( n \) models \( M_1, \ldots, M_n \), validation set \( V \), number of folds to perform cross-validation \( nf \), cross-validation function \( cv(M, V, nf) \)

**Ensure:** best model \( M^* \)

1. for \( i = 1 \) to \( n \) do
2. \( p[i] = cv(M_i, V, nf) \) \{array storing the performance of each model\}
3. end for
4. index = arg max\( i \) \( p[i] \) \{optimal index\}
5. \( M^* = M_{index} \) \{optimal model\}

---

The cross-validation function \( cv(M, V, nf) \) estimates the MDP and the historical policy using a training set obtained by partitioning the data \( V \) in \( nf \) equal-sized folds, and taking \( nf - 1 \) of these folds for training. The remaining fold is used for testing, to assess the log-likelihood of the derived Markov Chain or the prediction error on a fixed time horizon. The procedure is applied \( nf \) times, each time with a different test set, the average of all the performance measures is computed and returned by the function. I select the model with the best computed performance (highest log-likelihood or highest negative prediction error).

Once a model has been selected, I use bootstrap [26] to evaluate both the long-term value generated by the customers under the historical policy and the optimized long-term value generated by using the optimal policy found by backward induction.
3.6 Evaluation of marketing policies

Given a Markov Decision Process modeling the dynamics of customer relationships and a marketing policy, is it possible to derive the financial profile of each customer state induced by the policy. As MDPs model stochastic processes, Monte Carlo simulation enables the estimation of the range of possible lifetime values corresponding to a given customer state.

The Monte Carlo simulation is described in algorithm 3. The time window, the discount factor, and the number of simulations are parameters. \( \pi(a|s) \) is the marketing policy, i.e. the action probability distribution for each state. I randomly draw the reward using the inverse transformation method [49] on the estimated empirical cumulative distribution.

Algorithm 3 Monte Carlo simulation

Require: MDP model, policy \( \pi(a|s) \), time window \( T \), number of simulations \( n \), discount factor \( \lambda \)

Ensure: empirical value distribution \( dist[s] \) in each state

1: \( dist[s] = \{\} \) \{array of empirical distributions\}
2: \textbf{for all} \( s \in S \) \textbf{do}
3: \hspace{1em} \textbf{for} \( k = 1 \) to \( n \) \textbf{do}
4: \hspace{2em} \text{value} = 0
5: \hspace{2em} \textbf{for} \( t = 1 \) to \( T \) \textbf{do}
6: \hspace{3em} \( a \sim \pi(\cdot|s) \) \{draw randomly one action\}
7: \hspace{3em} \( r \sim r(\cdot|s,a) \) \{draw randomly the reward\}
8: \hspace{3em} \( s' \sim p(\cdot|s,a) \) \{draw randomly the next state\}
9: \hspace{3em} \text{value} = \text{value} + \lambda^{t-1}r
10: \hspace{2em} \textbf{end for}
11: \hspace{1em} \( dist[s] = dist[s] \cup value \)
12: \textbf{end for}
13: \textbf{end for}

Algorithm 3 captures the risk inherent in the customer relationship; the sources of this risk are the intrinsic stochastic behavior of customers and customer heterogeneity. In fact, customers in the same state can exhibit different behaviors leading to a range of possible lifetime values. In order to take the model parameter uncertainty into account, I combine the Monte Carlo simulation with bootstrap as described in algorithm 4.

Bootstrap is a general tool for assessing statistical accuracy [37]. Using bootstrap the MDP model is rebuilt several times, each time using a different training set. In this way the uncertainty in the model parameters is captured. Once the distribution of each state is known, the mean, the standard deviation, or the value-at-risk (see appendix A) can be computed and used for a risk-sensitive resource allocation (see chapter 4).
Algorithm 4 Bootstrap coupled with Monte Carlo simulation

Require: test set $Test$, policy $\pi(a|s)$, time window $T$, number of simulations $n$, discount factor $\lambda$, Monte Carlo simulation module $Sim(MDP, \pi(a|s), T, n, \lambda)$ returning an array of distributions for each state (algorithm 3), number of bootstrap samples $m$

Ensure: empirical value distribution $dist[s]$ in each state

1: $dist[s] = \{\}$ \{array of empirical distributions\}
2: $set = \{\}$ \{empty set\}
3: $c = |Test|$ \{$c$ is the cardinality of $Test$\}
4: for $k = 1$ to $m$ do
5: \begin{itemize}
6: \item extract randomly with replacement a subset $set$ of size $c$ from $Test$
7: \item estimate a Markov Decision Process $MDP$ from $set$
8: \item $dist[s] = dist[s] \bigcup Sim(MDP, \pi(a|s), T, n, \lambda)$ \{for each state merge the distribution computed with the simulation with the previous values\}
\end{itemize}
8: end for

3.7 Optimal marketing resource allocation

The optimal marketing policy is found by applying the backward induction algorithm 1 for a fixed time horizon. According to the customer’s current state, the optimal marketing policy suggests the action which maximizes the expected long-term value generated during the relationship. Heterogeneity across value-drivers, i.e. marketing actions, is therefore considered at a customer state (or segment) level. The expected customer equity $CE$ generated by the marketing policy is computed as follows:

$$CE = \sum_{i=1}^{n} CLV_i \cdot pop_i,$$
(3.12)

where $n$ is the number of customer states, $CLV_i$ is the average lifetime value in state $i$, and $pop_i$ is the current number of customers in state $i$. The improvement in customer equity $\triangle CE$ due to the optimal policy is easily computed as follows

$$\triangle CE = \sum_{i=1}^{n} \triangle CLV_i \cdot pop_i,$$
(3.13)

where $\triangle CLV_i$ is the average increment of lifetime value in state $i$. The increment is computed by evaluating the historical and the optimal policy.

I do not evaluate the impact of the marketing policy on future customers but restrict the analysis to the current customer base. If a model to forecast the growth of the customer base is built, formulas 3.12 and 3.13 can easily be updated to consider the value of the future customers.
In addition to the customer equity, it is possible to compute the investment required to implement the optimal marketing policy. In fact, by simulating the model and considering the expected costs generated for each state-action couple at the place of the expected rewards (i.e. the net cash flows), the future marketing investment $I$ needed to finance the marketing policy can be forecasted, i.e.

$$I = \sum_{i=1}^{n} cost_i \cdot pop_i,$$

where $cost_i$ is the expected long-term cost generated in state $i$. The return on investment $ROI$ due to a marketing policy is then computed as follows

$$ROI = \frac{CE}{I}.$$

Using bootstrap coupled with Monte Carlo simulation, it is possible to obtain confidence intervals for the customer equity $CE$, for the incremental lifetime values $\Delta CLV_i$ and for the incremental customer equity $\Delta CE$. The distribution of $\Delta CE$ can be used to test the hypothesis that the optimal marketing policy increments customer equity significantly, for instance the probability that $\Delta CE > 0$ can be estimated.

Finally, simulation enables the allocation of the marketing budget across customer states and time. This is especially important for corporate financial planning. For instance, the expected number of customers in each state in time can be forecasted using the Markov Chain resulting from the MDP and the marketing policy. Once the number of customers and the expected long-term investment in each state and at each time step are estimated, the total marketing budget can be allocated across time and customer states.

### 3.8 Case study

I apply the developed methodology to the customers of a major European airline and estimate the Markov Decision Process modeling the relationships with the company. Once the underlying Markov Decision Process has been estimated, I optimize the marketing policy by maximizing the lifetime value of the customer base.

#### 3.8.1 Data

I use transactional data of customers for a period length of two years, from January 2002 until December 2003. I reorganize the data into an episode structure similar to that defined in figure 3.3 where the time steps (i.e. decision epochs) are expressed in months. Each customer has therefore an associated episode containing a sequence of 24 events. I do not segment customers into a finite number of states, since this is part of the model definition.
Feature Description

<table>
<thead>
<tr>
<th>Feature</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>rectrip</td>
<td>elapsed time since last purchase</td>
</tr>
<tr>
<td>freqtrip3</td>
<td>number of transactions in the last 3 months</td>
</tr>
<tr>
<td>freqtrip12</td>
<td>number of transactions in the last 12 months</td>
</tr>
<tr>
<td>value3</td>
<td>value generated in the last 3 months</td>
</tr>
<tr>
<td>value3camp</td>
<td>value generated from responding to campaigns in the last 3 months</td>
</tr>
<tr>
<td>value12</td>
<td>value generated in the last 12 months</td>
</tr>
<tr>
<td>value12camp</td>
<td>value generated from responding to campaigns in the last 12 months</td>
</tr>
<tr>
<td>miles3</td>
<td>miles flown in the last 3 months</td>
</tr>
<tr>
<td>miles12</td>
<td>miles flown in the last 12 months</td>
</tr>
<tr>
<td>longevity</td>
<td>number of days since first transaction</td>
</tr>
</tbody>
</table>

Figure 3.5: Customer features.

At this stage each customer is represented with the set of numeric features defined in figure 3.5.

After removing the outliers\(^5\), I randomly extract 10000 customers. The customers are assigned randomly to two independent sets of size 5000: the validation set used in the model selection phase and the evaluation set used to optimize the marketing policy and to predict the future long-term value by simulating both the historical and the optimal policy.

**3.8.2 Defining customer states**

In order to build a MDP one needs to discretize the high-dimensional feature space into a finite number of states. I propose a list of segmentation criteria which can be divided into two categories: a) business-based segments, and b) statistical-based segments.

**Business-based segments**

The business-based segments are obtained by using recency, frequency, and monetary value. Each segmentation criterion can have several parameters. The segments are defined as follows.

- \(RFM(n)\) Scores the customers according to recency, frequency, and monetary value. The global customer score is determined first by the recency score, then by the frequency score, and finally be the monetary value score. The ranked customers are then

\(^5\)I removed marketing actions which have been applied very rarely and customers whose cumulative value is larger than the 99% percentile.
3.8. Case study

divided into \( n \) segments of equal size. Each segment is defined by an interval of values for the recency, the frequency, and the monetary value.

- **ABC\((a, b, c)\)** Scores the customers according to a value feature, e.g. value, and generates three segments by assigning the first \( a\% \) to segment A, the next \( b\% \) to segment B, and the remaining \( c\% \) to segment C. Customer in segment A and B are usually responsible for most of the total generated value.

- **VD\((a, b, c)\)** The Value-Defectors (VD) segmentation performs ABC\((a, b, c)\) segmentation both on a value feature and on a loyalty index, 9 segments are then obtained (e.g. AA, AB, AC, etc.).

- **RV\((a, b, c)\)** Recency-Value performs ABC\((a, b, c)\) segmentation both on a recency feature and on a value feature, there are 9 possible segments (e.g. AA, AB, AC, etc.).

**Statistical-based segments**

The following statistical-based segments use all the features defined in figure 3.5.

- **Trees\((n)\)** Regression Trees [16] are used for supervised clustering. A regression tree is trained to predict the immediate reward of each customer. The leaves of the tree correspond to the segments. The parameter \( n \) indicates the number of leaves in the training set obtained by acting on the parameters (i.e. the minimum size of a node) of the algorithm. This is a supervised clustering technique since the leaves are built specifically to minimize the standard deviation of the reward. Given a training set composed of input vectors \( x_i = (x_1, \cdots, x_m) \) and scalar outputs \( y_i \), regression trees partition recursively the input space into regions, for each region the average of the outputs is taken as prediction. The splitting is done so as to greedily minimize the sum of squared errors. For an input attribute \( x_j \) and a splitting point \( s \), the minimum mean squared error of the resulting binary split is given by

\[
\min_{c_1} \sum_{x_i \in R_1(x_j, s)} (y_i - c_1)^2 + \min_{c_2} \sum_{x_i \in R_2(x_j, s)} (y_i - c_2)^2
\]

where \( R_1(x_j, s) = \{x | x_j \leq s\} \) and \( R_2(x_j, s) = \{x | x_j > s\} \). \( c_1 \) and \( c_2 \) are constant values used as predictions in the two regions. It is possible to prove that the constant values minimizing the mean squared errors are the averages, i.e. \( c_1 = \mathbb{E}[y | y \in R_1] \) and \( c_2 = \mathbb{E}[y | y \in R_2] \). To find the best split, the mean squared error of each split and each feature is calculated, then the split achieving the smallest prediction error is selected.

\(^6\text{The loyalty index is a function of the frequency and the longevity of a customers and has been used by IBM in different CRM projects as a measure of the loyalty of customers.}\)
and the process iterates until a stopping criteria, such as some minimum node size is reached, is met. After, regression trees are pruned, i.e. those nodes which contribute the least to the total mean squared error are merged, the number of nodes to merge is usually found by cross-validation. Note that the input space is partitioned so as to minimize the prediction error and consequently the variance in each region.

- **SOM**($n, m$) Self-Organizing Maps [46] allow us to map the high-dimensional feature space into a two-dimensional rectangular $n \times m$ grid. The main idea is the following. Each cell in the rectangular map has an associated vector $w_i$ which has the same dimension as the feature space. In total there are $n \times m$ such vectors. The training is incremental, for each input $x$, the Euclidean distance between all the weights and $x$ is computed. Then the weight $w_j$ which is closer to $x$ is selected and all the weights $w_i$ which are in a neighbor of the winner $w_j$ are updated according to the Kohonen rule

$$w_i = w_i + \alpha(x - w_i),$$

where the learning rate $\alpha$ decreases slowly from 1 to 0. Neighbors are defined based on the position of the cells in the grid, I use all the cells which are adjacent to the winning cell. In practice, the Self-Organizing Map has been shown to be not sensitive to the particular definition of neighborhood. In order to not penalize features just because of the scale, I normalize all input features before applying the algorithm.

- **$K$-means**($n$) K-means clustering [37] finds $n$ clusters which are the centers minimizing the total within-cluster variance $V$ defined as

$$V = \frac{1}{2} \sum_{i=1}^{n} \sum_{C(m)=i} \sum_{C(m')=i} \|x_m - x_{m'}\|^2,$$

where $C(m)$ is the mapping that associates to the $m$-th training example $x_m$ a cluster (indexed from 1 to $n$) and the norm is computed using Euclidean distance. The following iterative algorithm can be used to minimize $V$: a) initialize randomly $n$ cluster centers, b) associate each data point to the nearest cluster center, c) recompute the clusters centers, d) repeat b) until some stopping condition is met. Usually the stopping condition is specified by the maximum number of possible iterations. Moreover, if the centers do not change the algorithm stops since a solution has been found. It can be shown that this algorithm always decreases the within cluster variance $V$, but convergence to a global minimum is not guaranteed. In practice the algorithm starts from different randomly chosen initial configurations and selects the solution with the lowest variance $V$. As in the case of SOM, I normalize all input features before applying the algorithm.
3.8.3 Model selection

I perform model selection by applying the algorithm 2 to the list of models defined in figure 3.7, each model is tested in the case of state-driven and action-driven approach using the validation set. When applying cross-validation the model is trained on the year 2002 and is tested on the year 2003 using a different customer set. In this way both the uncertainty due to applying the model to new customers and the uncertainty due to applying the model to predict the future is captured. Figure 3.8.3 summarizes the cross-validation process for the estimation of the mean prediction error.

Figure 3.6: Cross-validation allows us to estimate the distribution of the mean error by repeating the learning procedure $n$ times, each time a different fold is used as test set and the remaining folds are used for training the model.

As a first analysis I compare the mean absolute errors, computed with cross-validation, of each model using the state-driven and action-driven approaches. As shown in figure 3.8, the state-driven approach outperforms the action-driven approach for each segmentation.

Therefore I focus on the state-driven approach in the remainder of the chapter. The log-likelihood computed using cross-validation is shown in figure 3.9, the error bars represent the standard deviations of the mean log-likelihood and provide an estimate of the uncertainty of the estimation.

According to the log-likelihood, the best model is the $ABC$ value-based segmentation, the top-three models are #9, followed by #10 and #7. All the RFM segmentations perform very poorly. The main reason is due to the large number of states. It is harder to predict the future sequence of states if the number of possible states is large. Therefore, segmentations with a large number of states can be penalized by the likelihood criterion.

Figure 3.10 shows the mean absolute error with error bars. The performance of the RFM-based models is poor as well in this case. The best model is the regression tree with 10 leaves (#19), followed by $ABC$ (#10), the regression tree (#20), and $ABC$ (#9).
<table>
<thead>
<tr>
<th>#</th>
<th>Segment</th>
<th>Used features</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>RFM (10)</td>
<td>rectrip,freqtrip3,value3</td>
</tr>
<tr>
<td>2</td>
<td>RFM (20)</td>
<td>rectrip,freqtrip3,value3</td>
</tr>
<tr>
<td>3</td>
<td>RFM (30)</td>
<td>rectrip,freqtrip3,value3</td>
</tr>
<tr>
<td>4</td>
<td>RFM (10)</td>
<td>rectrip,freqtrip12,value12</td>
</tr>
<tr>
<td>5</td>
<td>RFM (20)</td>
<td>rectrip,freqtrip12,value12</td>
</tr>
<tr>
<td>6</td>
<td>RFM (30)</td>
<td>rectrip,freqtrip12,value12</td>
</tr>
<tr>
<td>7</td>
<td>ABC (10,10,80)</td>
<td>value3</td>
</tr>
<tr>
<td>8</td>
<td>ABC (10,20,70)</td>
<td>value3</td>
</tr>
<tr>
<td>9</td>
<td>ABC (10,10,80)</td>
<td>value12</td>
</tr>
<tr>
<td>10</td>
<td>ABC (10,20,70)</td>
<td>value12</td>
</tr>
<tr>
<td>11</td>
<td>VD (10,10,80)</td>
<td>value3,freqtrip3,rectrip</td>
</tr>
<tr>
<td>12</td>
<td>VD (10,20,70)</td>
<td>value3,freqtrip3,rectrip</td>
</tr>
<tr>
<td>13</td>
<td>VD (10,10,80)</td>
<td>value12,freqtrip12,rectrip</td>
</tr>
<tr>
<td>14</td>
<td>VD (10,20,70)</td>
<td>value12,freqtrip12,rectrip</td>
</tr>
<tr>
<td>15</td>
<td>RV (10,10,80)</td>
<td>value3,rectrip</td>
</tr>
<tr>
<td>16</td>
<td>RV (10,20,70)</td>
<td>value3,rectrip</td>
</tr>
<tr>
<td>17</td>
<td>RV (10,10,80)</td>
<td>value12,rectrip</td>
</tr>
<tr>
<td>18</td>
<td>RV (10,20,70)</td>
<td>value12,rectrip</td>
</tr>
<tr>
<td>19</td>
<td>Trees (10)</td>
<td>all</td>
</tr>
<tr>
<td>20</td>
<td>Trees (29)</td>
<td>all</td>
</tr>
<tr>
<td>21</td>
<td>K-means (10)</td>
<td>all</td>
</tr>
<tr>
<td>22</td>
<td>K-means (15)</td>
<td>all</td>
</tr>
<tr>
<td>23</td>
<td>K-means (20)</td>
<td>all</td>
</tr>
<tr>
<td>24</td>
<td>K-means (30)</td>
<td>all</td>
</tr>
<tr>
<td>25</td>
<td>SOM (3 × 3)</td>
<td>all</td>
</tr>
<tr>
<td>26</td>
<td>SOM (3 × 5)</td>
<td>all</td>
</tr>
<tr>
<td>27</td>
<td>SOM (4 × 5)</td>
<td>all</td>
</tr>
</tbody>
</table>

Figure 3.7: List of the used segmentations.
Figure 3.8: Comparison of the mean absolute error based on the prediction of the value generated in 12 months, for action-driven and state-driven approach.

The tree-based clustering seems to provide a good tradeoff between mean absolute error and likelihood. I select this segmentation in order to model the dynamics of customer relationships.

By examining figure 3.10 one can conclude that:

- the RFM segmentation which has been largely used in the literature [18], [32], [10], [67], is not necessarily the best way to model customer dynamics,
- statistical-based segmentation criteria such as regression trees and self-organizing maps, outperform most of the segmentation criteria based on frequency and recency.

### 3.8.4 Simulation and maximization of customer lifetime value

I use an independent data set (evaluation set) to train the MDP using the tree-based segmentation #19. Then I estimate the historical policy and simulate the Markov Decision Process in order to predict the future value obtained by following the historical policy. The Monte Carlo simulation is described in algorithm 3.

In order to maximize the long-term value, I apply the backward induction defined by the algorithm 1 and find the optimal policy. For confidentiality reasons I cannot show the
historical policy and the optimal policy in terms of marketing actions undertaken by the company. Now I can apply the Monte Carlo simulation coupled with bootstrap defined in algorithm 4.

Figure 3.11 compares the expected long-term value obtained by simulating the Markov Decision Process for 12 months with 30 bootstrap samples using the optimal and the historical policy. The optimal policy obtained using dynamic programming outperforms the short-sighted historical policy according to the model simulations.

Figures 3.12 and 3.13 show, respectively, the expected long-term value and the error bars generated using the historical and the optimal policy. Analyzing the error bars, one can note that the optimal policy leads to higher expected values but as well to higher uncertainties in the predictions.

3.9 Conclusions and managerial implications

In this chapter I provide a rigorous methodology for the estimation of robust Markov Decision Processes modeling the dynamic relationship between the customers and the company. I use a non-parametric approach to estimate the model, i.e. the transition probabilities and the rewards given a state and an action.

Although the use of Markov Decision Processes and dynamic programming is not new in the marketing and the management science literature, the issue of estimating robust mod-
3.9. Conclusions and managerial implications

Figure 3.10: Mean absolute error with error bars (± standard deviation of the mean).

elements from customer relationship transactional data has not been addressed so far. As shown in [82], the optimal marketing policy can be sensitive to the Markov Decision Process estimation. In fact, if transition probabilities and reward estimates are not reliable, the optimal value function can be biased leading to a policy that can perform worse than the historical policy in certain states. The lack of reliability can be simply due to the small amount of data used to estimate the model.

As the number of states affects the amount of data used to estimate the parameters, it can have an impact on the reliability of the estimated MDP. Moreover, the feature discretization methodology can dramatically affect the performance of the model, as revealed in the case study. Using cross-validation for model selection, I build a reliable Markov Decision Process that takes into account the impact that the uncertainty in the parameters of the model has on the performance measure. The approach developed enables an efficient allocation of marketing resources by optimizing the long-term return on investment using the optimal marketing policy. Lifetime value is considered as an endogenous variable depending on both the customer state and the marketing action. Customer heterogeneity across states (i.e. segments) and marketing action heterogeneity are addressed by tailoring the marketing action to the individual customer.

In a case study I show that the popular RFM segmentation does not lead to the best model both in terms of likelihood and in terms of prediction error. Statistical segmentation criteria, such as regression trees and self-organizing maps, or simple criteria such as considering
Figure 3.11: Comparison of the long-term value generated in 12 months when using the optimal and the historical policy.

the value generated in the last twelve months, lead to models that outperform the RFM segmentation.

Finally, the risk inherent both in the stochastic nature of the customer relationship and in the model uncertainty can be estimated explicitly using bootstrap as shown in figures 3.12 and 3.13. As the distributions of the future return on investment can be estimated, the model enables risk-sensitive resource allocation.
3.9. Conclusions and managerial implications

Figure 3.12: Value generated in each state using the historical policy (± standard deviation of the mean).

Figure 3.13: Value generated in each state using the optimal policy (± standard deviation of the mean).
Chapter 4

Customer Portfolios

4.1 Introduction

In this chapter I discuss how to build customer portfolios maximizing expected customer equity and minimizing the risk while satisfying constraints on the available marketing budget. This chapter assumes that the individual lifetime value distributions have been estimated using either the model described in chapter 3 or historical simulation [42]. In the case of historical simulation, the marketing policy is not optimized since the history of customer transactions is used to estimate the distributions. The advantage of using Markov Decision Process models coupled with Monte Carlo simulation is that the lifetime value distributions induced by the optimal marketing policy can be estimated.

As noted in chapter 1, the past and current marketing literature (e.g. [79], [78], [11], [12], [41], [33], [34], [67]) has focused on the allocation of marketing resources to optimize the expected customer equity or the expected customer lifetime value. However, if marketing investments are evaluated from a financial perspective, as for example in [22] and [83], then the risk of the investment should be quantified and managed as in common financial practice [54], [15].

After determining the distribution of the customer equity as a function of the marketing budget, I formulate the optimization problem defining the customer portfolio with the maximum expected value and the minimum risk. I use the optimization techniques and concepts described in appendix A.

When evaluating the risk profile of customer assets, there is not always an effective value-risk tradeoff. In contrast, in the case of financial assets, the value-risk tradeoff is induced by the laws of supply and demand governing the nearly efficient markets. I discuss the conditions under which a value-risk tradeoff is a relevant issue. For instance, uncertainty due to the model parameters can have a large impact on the value-risk tradeoff.
The chapter is organized as follows. In section 4.2 I formulate the research problem and introduce the notation. In section 4.3 I describe an approximation to estimate the distribution of the total customer equity. In sections 4.4 and 4.5 I discuss how to allocate the marketing budget by formulating and solving the associated mathematical constrained optimization problem. In section 4.6 I provide some examples of marketing budget allocation using the developed model. In section 4.7 I discuss how the parameter uncertainty affects the distribution of the customer equity. In sections 4.8, 4.9, and 4.10 I focus on the value-risk tradeoff in the case of customer portfolios. Finally, the conclusions and managerial implications follow in section 4.11.

4.2 Problem formulation

For each customer in a given state (i.e. customer segment) the following distributions are assumed to be known:

- the distribution of the total revenue generated during a given time period,
- the distribution of the total cost generated during a given time period,
- the distribution of the total return on marketing investment (ROI) generated during a given time period.

Given $N$ customer segments, I define:

- $\max(i)$ as the maximum number of customers that can be targeted in segment $i$. This number is equal to the number of customers currently in the segment.
- $\beta$ as the overall available marketing budget.
- $c_i(j)$ as the cost distribution generated by targeting customer $j$ in segment $i$ in a given time period.
- $r_i(j)$ as the revenue distribution generated by targeting customer $j$ in segment $i$ in a given time period.
- $p_i(j)$ as the profit distribution generated by targeting customer $j$ in segment $i$ in a given time period, note that $p_i(j) = r_i(j) - c_i(j)$.
- $roi_i(j)$ as the return on investment distribution generated by targeting customer $j$ in state $i$ in a given time period, note that $roi_i(j) = \frac{p_i(j)}{c_i(j)}$.

\footnote{We use the concepts of distribution and random variable interchangeably here.}
• $n_i$ number of customers to be targeted in segment $i$. This is the decision variable of the marketing budget allocation problem.

I assume that these distributions are independent and identically distributed (i.i.d.) given the segment. This assumption is motivated by the following considerations: a) customers represent different physical people therefore the reactions of two randomly selected customers to a given marketing campaign can be assumed to be independent, and b) the segmentation is supposed to gather different customers which behave in a similar way, therefore one can assume customers in one segment have the same distribution. Moreover, I assume that the distributions of different segments are independent. Eventual correlations are therefore ignored as they are in general not measurable given the historical marketing data, i.e. the costs and revenues generated by targeting the customers. The value\(^2\) generated by targeting $n$ customers in segment $i$ is:

$$value_i(n) = \sum_{k=1}^{n} p_i(k) = \sum_{k=1}^{n} roi_i(k) \cdot c_i(k). \quad (4.1)$$

The cost generated by targeting $n$ customers in segment $i$ is:

$$cost_i(n) = \sum_{k=1}^{n} c_i(k). \quad (4.2)$$

Note that both the value and the cost are random variables with a given probability density function.

**Definition 4.2.1 (Customer Equity).** The customer equity ($CE$) is equal to the sum of the value of the segments, given the number of customers to target in each segment\(^3\):

$$CE(n_1, n_2, \cdots, n_i, \cdots, n_N) = \sum_{i=1}^{N} value_i(n_i) = \sum_{i=1}^{N} \sum_{k=1}^{n_i} roi_i(k) \cdot c_i(k) \quad (4.3)$$

where $n_i$ is the number of customers targeted in segment $i$.

The optimization problem that I address is the following.

**Problem 4.2.1.** How many customers in each segment should be targeted with a limited marketing budget in order to optimize the value of the customer equity?

An alternative formulation of the above research question is the following.

---

\(^2\)The terms “value” and “profit” are used interchangeably in this chapter.

\(^3\)Here I consider the customer equity as the value generated by targeting the customer base over a given time interval and with a given set of marketing actions. Others [79] define the customer equity as the cumulative value generated over a given time interval by all the current and future customers of the company.
Problem 4.2.2. Given a limited marketing budget, how much should be invested in each customer segment in order to optimize the customer equity?

The objective is therefore to build a portfolio of customers which are selected from different segments. Each segment represents an asset category. In order to find the optimal budget allocation one needs to define a criterion for evaluating the final portfolio distribution $CE$. Moreover, as the total cost is, generally, modeled as a random variable, it is necessary to define probabilistic constraints. Optimizing the customer equity by finding the optimal number of customers to target in each segment is therefore a stochastic optimization problem. The following section describes an approximation to derive the probability distribution of the customer equity.

4.3 The normal density approximation

As described in appendix A, both the risk management and the utility-based approach consider the uncertainty in the risk factors when optimizing an investment. In the risk management approach, these uncertainties are modeled by a stochastic loss function. A risk measure mapping the loss distribution to a real value is then minimized. According to the expected utility approach, after defining the utility function of the decision maker, the expected utility of the gain is maximized. In order to compute the expected utility the distribution of the gain function (or the underlying risk factors) needs to be known.

The first issue to solve in order to allocate a marketing budget for the customer base is therefore the determination of the probability density function of the customer equity as defined in equation (4.3), which can be interpreted either as a gain function $G = CE$, or as a loss function $L = -CE$. According to section 4.2, the customer equity has been defined as follows:

$$CE(n_1, n_2, \ldots, n_i, \ldots, n_N) = \sum_{i=1}^{N} value_i(n_i).$$

Let us first focus on the distribution of the total value for a given segment. The value given by targeting $n$ customers of segment $i$ is:

$$value_i(n) = \sum_{k=1}^{n} roi_i(k) \cdot c_i(k) = \sum_{k=1}^{n} p_i(k).$$

The random variable $value_i$ depends therefore on the number of targeted customers $n$ in a non-linear way. Under the assumption that the random variables $p_i(k)$ are independent and identically distributed, with mean and variance $\mu_i$ and $\sigma_i^2$ respectively, the Central Limit Theorem [60] implies that for large values of $n$:

$$\frac{1}{n} \sum_{k=1}^{n} p_i(k) \sim \mathcal{N}(\mu_i, \frac{\sigma_i^2}{n})$$
4.3. The normal density approximation

where \( \mathcal{N}(\mu_i, \sigma^2_i) \) is the normal distribution with mean \( \mu_i \) and variance \( \sigma^2_i \). The above formula implies that:

\[
\text{value}_i(n) \sim \mathcal{N}(n\mu_i, n\sigma^2_i).
\]  

For large values of \( n \), the value distribution of segment \( i \) approximates therefore a normal distribution with mean \( n\mu_i \) and variance \( n\sigma^2_i \). The Central Limit Theorem does not make any hypothesis about the distributions of the random variables which are summed. In practice, the convergence to the normal distribution depends on the shape of the distributions of the random variables being summed, asymmetric distributions or distributions with heavy tails converge slower. Given the observed data, the minimum \( n \) in order to obtain an approximation to the normal distribution can be determined by starting with a given value and then testing the hypothesis that the sum of random variables is a normal distribution. If the test fails, \( n \) is increased until the hypothesis cannot be rejected. In many practical cases a value of \( n \geq 30 \) is sufficient for a good approximation of a normal distribution [57].

By extending the normal approximation to the other segments, the customer equity is given by a sum of normally distributed random variables. As the sum of independent normally distributed random variables is a normally distributed random variable [60], the probability density function of the customer equity as a function of the decision parameters \( n_1, \cdots, n_N \) is

\[
CE(n_1, \cdots, n_N) = \sum_{i=1}^{N} \text{value}_i(n_i) \sim \mathcal{N}\left(\sum_{i=1}^{N} n_i\mu_i, \sum_{i=1}^{N} n_i\sigma^2_i\right).
\]  

Equation (4.5) shows that if for each customer segment a large number of customers is targeted, then the total value generated by the marketing investment is approximately distributed as a normal distribution having mean:

\[
\mu_{CE} = \sum_{i=1}^{N} n_i\mu_i
\]  

and variance:

\[
\sigma^2_{CE} = \sum_{i=1}^{N} n_i\sigma^2_i.
\]

Here I assume to know the mean \( \mu_i \) and the variance \( \sigma^2_i \) of the profit distribution \( p_i \) of segment \( i \). A simple way to model the future profit distributions for the customer segments is to assume that the observed historical distributions do not change in the future (e.g. history-based simulation). In this case, provided that historical data is available, the mean and the variance of each segment can be estimated from the data as follows:

\[
\mu_i = \frac{1}{n} \sum_{k=1}^{n} p_i(k)
\]

\[4\] Here I assume that the distributions of different segments are independent.
\[ \sigma_i^2 = \frac{1}{n-1} \sum_{k=1}^{n} (p_i(k) - \mu_i)^2. \]

Assuming that a large amount of data is available, the estimates for each segment will be reliable.

By approximating the customer equity with a normal distribution, one can use the mean-variance framework to minimize the risk for a given level of expected value. As the normal distribution is symmetric, the variance properly measures the risk. Moreover, as will be shown in the next sections, in the case of a loss (or gain) function that is normally distributed, minimizing the Value-at-Risk or the Conditional Value-at-Risk is the same as minimizing the variance.

In order to apply the Central Limit Theorem, one has to assume that a large number of customers are targeted in each segment. This assumption is quite realistic in practice. In fact, many companies are using CRM technologies allowing to store and analyze the data of thousands, if not millions, of customers. The normal approximation is therefore well suited in a database-marketing context.

### 4.3.1 VaR for normal loss distribution

If the cumulative loss distribution \( F_L \) is normal with mean \( \mu_L \) and variance \( \sigma_L^2 \) the Value-at-Risk for a given confidence level \( \alpha \in (0, 1) \) is given by:

\[
VaR_{\alpha}(L) = \mu_L + \sigma_L \Phi^{-1}(\alpha)
\]

(4.8)

where \( \Phi(\cdot) \) is the standard normal\(^5\) cumulative density function, \( \Phi^{-1}(\alpha) \) is therefore the \( \alpha \)-quantile. In fact, as the Value-at-Risk is by definition the \( \alpha \)-quantile of a continuous cumulative density function (see appendix A), we obtain:

\[
F_L(VaR_{\alpha}) = \alpha
\]

but the left hand of the above formula is also equal to:

\[
F_L(VaR_{\alpha}) = P(L \leq VaR_{\alpha}) = P\left( \frac{L - \mu_L}{\sigma_L} \leq \frac{VaR_{\alpha} - \mu_L}{\sigma_L} \right) = \Phi\left( \frac{VaR_{\alpha} - \mu_L}{\sigma_L} \right)
\]

therefore

\[
\Phi\left( \frac{VaR_{\alpha} - \mu_L}{\sigma_L} \right) = \alpha
\]

which implies that

\[
\frac{VaR_{\alpha} - \mu_L}{\sigma_L} = \Phi^{-1}(\alpha) \rightarrow VaR_{\alpha} = \mu_L + \sigma_L \Phi^{-1}(\alpha).
\]

\(^5\)The standard normal distribution has zero mean and unitary variance.
4.3. The normal density approximation

Minimizing the Value-at-Risk for a given level of expected value \( \hat{\mu}_L \) and a given confidence level \( \alpha \) is therefore equivalent to minimizing the variance \( \sigma_L^2 \) of the normally distributed loss function.

4.3.2 CVaR for normal loss distribution

If the cumulative loss distribution \( F_L \) is normal with mean \( \mu_L \) and variance \( \sigma_L^2 \), the Conditional Value-at-Risk for a given confidence level \( \alpha \in (0, 1) \) is given by:

\[
CVaR_\alpha(L) = \mu_L + \sigma_L \varphi(\Phi^{-1}(\alpha)) \frac{1}{1 - \alpha}
\]

(4.9)

where \( \Phi(\cdot) \) is the standard normal cumulative density function and \( \varphi \) is the standard normal probability density function. In fact, according to formula (A.12) in appendix A, the Conditional Value-at-Risk is defined as:

\[
CVaR_\alpha(L) = \mathbb{E}[L | L \geq VaR_\alpha(L)].
\]

As the loss \( L \) is normally distributed, it can be rewritten as:

\[
L = \mu_L + \sigma_L X
\]

where the random variable \( X \) has a standard normal distribution, i.e. \( X \sim \mathcal{N}(0, 1) \). By writing \( L \) as a function of \( X \) we obtain:

\[
CVaR_\alpha(\mu_L + \sigma_L X) = \mathbb{E}[\mu_L + \sigma_L X | \mu_L + \sigma_L X \geq VaR_\alpha(L)].
\]

For equation (4.8) we deduce that the right hand of the above equation is equal to:

\[
\mathbb{E}[\mu_L + \sigma_L X | \mu_L + \sigma_L X \geq \mu_L + \sigma_L \Phi^{-1}(\alpha)] = \mathbb{E}[\mu_L + \sigma_L X | X \geq \Phi^{-1}(\alpha)].
\]

By definition of conditional expectation, the right hand of the above equation is equal to:

\[
\frac{\int_{\Phi^{-1}(\alpha)}^{+\infty} (\mu_L + \sigma_L x) \varphi(x) dx}{\int_{\Phi^{-1}(\alpha)}^{+\infty} \varphi(x) dx} = \frac{\mu_L (1 - \alpha) + \sigma_L \int_{\Phi^{-1}(\alpha)}^{+\infty} x \varphi(x) dx}{1 - \alpha}.
\]

(4.10)

Given that for the standard normal probability density function \( \varphi(x) \) the following property holds:

\[
\varphi'(x) = -x \varphi(x)
\]

the integral in equation (4.10) is equal to:

\[
\int_{\Phi^{-1}(\alpha)}^{+\infty} x \varphi(x) dx = -\int_{\Phi^{-1}(\alpha)}^{+\infty} \varphi'(x) dx = \varphi(\Phi^{-1}(\alpha)) - \varphi(+\infty) = \varphi(\Phi^{-1}(\alpha)).
\]

Solving the integral in equation (4.10) gives formula (4.9). Minimizing the Conditional Value-at-Risk for a given level of expected value \( \hat{\mu}_L \) and a given confidence level \( \alpha \) is therefore equivalent to minimizing the variance \( \sigma_L^2 \) of the normally distributed loss function.
Chapter 4. Customer Portfolios

4.4 Optimizing customer portfolios

The optimal customer portfolio which maximizes the expected value and minimizes the risk is given by solving the following problem:

\[
\min \sigma_{CE}^2 \\
\mu_{CE} \geq \hat{\mu}
\] (4.11)

By changing the threshold \( \hat{\mu} \) the efficient frontier is obtained. Assuming the different segments are not correlated, one can use formulas (4.6) and (4.7), thus the above problem becomes:

\[
\min \sum_{i=1}^{N} n_i \sigma_i^2 \\
\sum_{i=1}^{N} n_i \mu_i \geq \hat{\mu}_{CE}
\] (4.12)

where the decision variables \( n_1, \ldots, n_N \) are integers. Problem (4.12) is therefore a linear integer program (IP). Integer programs are in general much harder\(^6\) to solve than linear programs (LP). To avoid dealing with integer programs, I simply remove the restriction that the decision variables need to be integers and the problem becomes a standard linear program. Once the solutions are found, I round the real numbers to integers. In general, this method of dealing with integer programs does not guarantee finding the optimal solution. In this case, however, the solution will be nearly optimal. In particular, the hypothesis, leading to the normal distribution approximation, assumes that a large number of customers for each segment is targeted. In this case, the total customer equity is much larger than \( p_i \), which is the contribution of an individual customer of segment \( i \). Therefore the error introduced by rounding a real number to an integer has a very small impact in terms of expected value and of risk.

Problem (4.12) does not consider any constraints. As discussed in section 4.2, there are two types of constraints in the customer equity optimization problem:

1. for each customer segment \( i \), not less than zero and no more than \( \max(i) \) customers can be targeted,

2. a limited marketing budget \( \beta \) is available, whereby the total costs cannot exceed this budget.

\(^6\)Integer programs are in general NP-hard which means that there are not algorithms able to solve them in linear or polynomial time. Large dimensional NP problems are usually not solvable given reasonable time constraints because they grow exponentially.
4.4. Optimizing customer portfolios

The first constraint is analytically modeled as follows:

\[ 0 \leq n_i \leq \max(i) \quad i : 1, \cdots, N. \quad (4.13) \]

Since the cost of targeting a customer is modeled as a random variable, the total cost to target customers in the different segments is also a random variable and is equal to:

\[ \text{cost}_{CE} = \sum_{i=1}^{N} \text{cost}_i(n_i) = \sum_{i=1}^{N} \sum_{k=1}^{n_i} c_i(k) \]

where \( \text{cost}_i(n_i) \) is the total cost generated by targeting \( n_i \) customers of segment \( i \); \( c_i(k) \) is the cost incurred by targeting customer \( k \) in segment \( i \). As in the case of the customer equity (4.3), the total cost is a random variable. In order to define a constraint on the cost, one needs therefore to define a risk measure to minimize. An appropriate risk measure is the Value-at-Risk (VaR) for a given percentile level \( \alpha \). The constraint

\[ \text{VaR}_\alpha(\text{cost}_{CE}) \leq \beta \quad (4.14) \]

means that there is a probability of \((1 - \alpha)\) of exceeding the available marketing budget. The above constraint is equivalent to the following stochastic constraint:

\[ P(\text{cost}_{CE} \geq \beta) \leq 1 - \alpha \]

in fact:

\[ P(\text{cost}_{CE} \geq \beta) \leq 1 - \alpha \rightarrow 1 - P(\text{cost}_{CE} \leq \beta) \leq 1 - \alpha \rightarrow P(\text{cost}_{CE} \leq \beta) \geq \alpha \]

\[ \rightarrow F_{\text{cost}_{CE}}(\beta) \geq \alpha \]

where \( F_{\text{cost}_{CE}} \) is the cumulative density function of the total cost, since the cumulative function is monotonically increasing, the above formula implies that

\[ F_{\text{cost}_{CE}}^{-1}(F_{\text{cost}_{CE}}(\beta)) \geq F_{\text{cost}_{CE}}^{-1}(\alpha) \]

which means that

\[ \beta \geq F_{\text{cost}_{CE}}^{-1}(\alpha). \]

As the right hand of the above equation is the \( \alpha \)-quantile of the loss distribution, it is by definition equal to \( \text{VaR}_\alpha(\text{cost}_{CE}) \); therefore the equivalence is proven.

The next step is to find the distribution of the total targeting cost. As in the case of the value distribution, one can assume that the individual costs for a given segment \( i \), i.e. \( c_i(k) \), are independent and identically distributed (i.i.d.). According to the Central Limit Theorem, the sum of the individual costs, i.e. the total cost generated by targeting a given
segment, will be approximately normally distributed if the number of targeted customers is large. As the sum of normal distributions is a normal distribution, the total costs generated by targeting all the segments will be approximately normally distributed. Applying the same line of reasoning used in the case of the value distributions, one can deduce that the cost distribution is:

\[
\text{cost}_{CE} \sim \mathcal{N}(\sum_{i=1}^{N} n_i \mu_{c,i}, \sum_{i=1}^{N} n_i \sigma_{c,i}^2)
\]

(4.15)

where \(\mu_{c,i}\) and \(\sigma_{c,i}^2\) are respectively the mean and the variance of the cost distribution in segment \(i\). These quantities can be estimated from the available data.

Using formula (4.8), under the normal approximation, the Value-at-Risk is:

\[
\mu_{cost_{CE}} + \sigma_{cost_{CE}} \Phi^{-1}(\alpha)
\]

where

\[
\mu_{cost_{CE}} = \sum_{i=1}^{N} n_i \mu_{c,i}
\]

and

\[
\sigma_{cost_{CE}} = \sqrt{\sum_{i=1}^{N} n_i \sigma_{c,i}^2}
\]

Constraint (4.14) is therefore equal to the following inequality:

\[
\sum_{i=1}^{N} n_i \mu_{c,i} + \Phi^{-1}(\alpha) \sqrt{\sum_{i=1}^{N} n_i \sigma_{c,i}^2} \leq \beta.
\]

(4.16)

Under normal approximation, the customer equity optimization problem is therefore:

\[
\min_{n_1, \ldots, n_N} \sum_{i=1}^{N} n_i \sigma_i^2
\]

\[
\sum_{i=1}^{N} n_i \mu_i \geq \hat{\mu}_{CE}
\]

(4.17)

\[
\sum_{i=1}^{N} n_i \mu_{c,i} + \Phi^{-1}(\alpha) \sqrt{\sum_{i=1}^{N} n_i \sigma_{c,i}^2} \leq \beta
\]

\[0 \leq n_i \leq \max(i) \quad i : 1, \ldots, N.
\]

Problem (4.17) is not a convex optimization problem since the constraint (4.16) is not convex, in fact it is possible to prove that constraint (4.16) defines a concave function if \(\Phi^{-1}(\alpha)\) is positive. This is the case since \(\alpha > 0.5\). The proof is based on showing that the constraint defines a concave function, as a consequence the inequality defines a concave set. The concavity of the constraint function can be shown by verifying that the associated hessian matrix
is not positive definite, or by restricting the function to a line and showing the concavity of a function of only one variable [14] by computing the second derivative.

Therefore the region identified by the inequality is in general not convex; this fact implies that a local optima is not always a global optima. Problem (4.17) can be solved with the following approximation.

4.4.1 Linear approximation

According to the assumption that for each segment a large number of people will be targeted, one can ignore the standard deviation and write:

$$\mu_{\text{cost}_{CE}} + \sigma_{\text{cost}_{CE}} \Phi^{-1}(\alpha) \simeq \mu_{\text{cost}_{CE}}. \quad (4.18)$$

The above approximation is justified by the following considerations. If the decision variables $n_i$ are large, the following inequality holds:

$$\left( \sum_{i=1}^{N} n_i \mu_{c,i} \right)^2 > \sum_{i=1}^{N} n_i \sigma_{c,i}^2,$$

as the left hand of the above inequality is quadratic in the variables $n_i$ while the right hand is linear; moreover, $\mu_{c,i} > 0$ as costs are positive by definition. The above inequality is motivated by the fact that

$$\lim_{n_1, \ldots, n_N \to +\infty} \frac{\sum_{i=1}^{N} n_i \sigma_{c,i}^2}{\left( \sum_{i=1}^{N} n_i \mu_{c,i} \right)^2} = 0.$$

The $\alpha$-quantile of the standard normal distribution is approximately equal to 2 for large values of $\alpha$, therefore the above inequality implies that

$$\sum_{i=1}^{N} n_i \mu_{c,i} > \Phi^{-1}(\alpha) \sqrt{\sum_{i=1}^{N} n_i \sigma_{c,i}^2}.$$

Another argument, justifying the approximation (4.18) is the following. For large values of the decision variables, the average cost $\mu_{\text{cost}_{CE}}$ is much more sensitive to changes in $n_i$ than the standard deviation $\sigma_{\text{cost}_{CE}}$. This can be seen with a sensitivity analysis, considering the partial derivatives of the mean and the standard deviation:

$$\frac{\partial \mu_{\text{cost}_{CE}}}{\partial n_i} = \mu_{c,i},$$

$$\frac{\partial \sigma_{\text{cost}_{CE}}}{\partial n_i} = \frac{\sigma_{c,i}^2}{\sqrt{\sum_{i=1}^{N} n_i \sigma_{c,i}^2}}.$$
The sensitivity of the standard deviation tends to zero if the quantity of targeted customer is large:

$$\lim_{n_1, \ldots, n_N \to +\infty} \frac{\partial \sigma_{cost_{CE}}}{\partial n_i} = 0.$$ 

The squared root in constraint (4.16) will have a very small impact on the solution and can be ignored. An indirect consequence of the hypothesis leading to the application of the Central Limit Theorem is therefore that problem (4.17) can be approximated by the following linear program:

$$\min_{n_1, \ldots, n_N} \sum_{i=1}^{N} n_i \sigma_i^2$$

$$\sum_{i=1}^{N} n_i \mu_i \geq \hat{\mu}_{CE}$$

$$\sum_{i=1}^{N} n_i \mu_{c,i} \leq \beta$$

$$0 \leq n_i \leq \max(i) \quad i : 1, \ldots, N.$$  

Problem (4.19) is an approximation in the case that the costs are stochastic, but it is the exact optimization problem in the case that the costs are deterministic. In fact, in the deterministic case, if the cost to target segment $i$ is indicated with $\mu_{c,i}$ the budget constraint is

$$\sum_{i=1}^{N} n_i \mu_{c,i} \leq \beta.$$ 

Another interpretation of problem (4.19) is the following. If costs are random variables, they can be decomposed into a deterministic component $\mu_{c,i}$ and a random component $\epsilon_i$. Therefore the cost to target one customer in segment $i$ is

$$c_i(k) = \mu_{c,i} + \epsilon_i.$$ 

By definition, the profit obtained by targeting a given customer $k$ in segment $i$ is

$$p_i(k) = r_i(k) - c_i(k) = r_i(k) - \epsilon_i - \mu_{c,i}.$$ 

therefore the stochastic component of the cost can be incorporated into the revenue $r_i'(k)$ without loss of generalization, and the cost $c_i'(k)$ is deterministic. This formulation of the problem allows us to consider costs as deterministic and minimize the risk of the customer equity for a given expected value. If the efficient customer portfolio is formulated as in problem (A.8), and the normal approximation for the customer equity is considered, the minimization of the Value-at-Risk, the variance, the Conditional Value-at-Risk are equivalent.
for every confidence level $\alpha$. In fact, as previously shown, both the VaR and the CVaR are only functions of the standard deviation if the confidence level $\alpha$ and the mean $\hat{\mu}_{CE}$ are fixed, this leads to the following optimization problem:

$$
\begin{align*}
\min_{n_1, \ldots, n_N} & \sum_{i=1}^{N} n_i \sigma_i^2 \\
\sum_{i=1}^{N} n_i \mu_i &= \hat{\mu}_{CE} \\
\sum_{i=1}^{N} n_i \mu_{c,i} &\leq \beta \\
0 &\leq n_i \leq \max(i) \quad i : 1, \cdots, N.
\end{align*}
$$

Problem (4.20) differs from problem (4.19) only for the inequality constraint on the expected value which is substituted with an equality constraint. Moreover, problem (4.20) is not an approximation, but represents the exact solution of the customer equity optimization problem if the constraint on the risk is defined on the total value and not on the total cost and if the constraint on the expected customer equity is an equality constraint and if the Central Limit Theorem can be applied. This formulation of the problem is particularly advantageous, since the optimization problem is linear and the only approximation is the normal distribution which has a strong theoretical motivation if many customers are targeted. Note that, without considering the constraint on the budget, the constraint $\hat{\mu}_{CE}$ can only assume values in the interval

$$
\hat{\mu}_{CE} \in \left[0, \sum_{\mu_i \geq 0} \max(i) \mu_i \right]
$$

since the worst expected value of the customer portfolio is 0, assuming that investors do not select a portfolio with negative expected value, and the possible highest value is given by targeting all the customers in the segments with positive expected profit. By varying the expected customer equity $\hat{\mu}_{CE}$ the efficient frontier of the customer portfolio is obtained.

## 4.5 Customer portfolio with correlations

So far I discussed the normal density approximation in the case that the costs and the revenues generated by different customer segments were not correlated. In this section I consider correlation within segments. Two customer segments can be correlated in terms of their profit distributions because of correlations between macro-economic indicators of two different geographic regions which are in turn correlated to the buying behavior of customers.

Assuming one can measure the correlation $\rho_{i,j}$ between segments $i$ and $j$ and that the
customer equity is normally distributed\(^7\), then

\[
CE(n_1, \ldots, n_N) = \sum_{i=1}^{N} value_i(n_i) \sim N\left(\sum_{i=1}^{N} n_i \mu_i, \sum_{i=1}^{N} \sum_{j=1}^{N} \sqrt{n_i n_j} \sigma_i \sigma_j \rho_{i,j}\right).
\]

Using the same line of reasoning that led to problem (4.20), the optimum portfolio in the case of correlated segments is given by the solution of the following optimization problem:

\[
\begin{align*}
\min_{n_1, \ldots, n_N} & \sum_{i=1}^{N} \sum_{j=1}^{N} \sqrt{n_i n_j} \sigma_i \sigma_j \rho_{i,j} \\
& \sum_{i=1}^{N} n_i \mu_i = \hat{\mu}_{CE} \\
& \sum_{i=1}^{N} n_i \mu_{c,i} \leq \beta \\
& 0 \leq n_i \leq \text{max}(i) \quad i : 1, \ldots, N.
\end{align*}
\]

Unfortunately the above optimization problem is not a convex. In fact, by introducing the variable \(m_i = \sqrt{n_i}\), the objective function becomes a quadratic form but the linear equality constraint becomes

\[
\sum_{i=1}^{N} m_i^2 \mu_i = \hat{\mu}_{CE},
\]

this constraint does not define a convex set \([14]\), therefore the equivalent problem obtained by introducing the variables \(m_i\) is not convex. To solve this problem optimization heuristics such as genetic algorithms \([31]\), \([58]\), can be used.

In general, the correlations between segments will depend on the number of customers that are in each segment. Problem (4.21) assumes that \(\rho_{i,j}\) does not depend on the decision variables \(n_1, \ldots, n_N\). Next I reformulate the optimization problem by considering a possible analytical form for the correlations. Assuming that one can measure the covariance \(\sigma_{i,j}\) between the average profit per customer for segment \(i\) and segment \(j\), one obtains:

\[
\text{cov}(value_i(n_i), value_j(n_j)) = \text{cov}(n_i \cdot av_i, n_j \cdot av_j) = n_i n_j \text{cov}(av_i, av_j) = n_i n_j \sigma_{i,j}
\]

where \(\text{cov}(\cdot, \cdot)\) is the covariance between two random variables and \(av_i\) is the average profit per customer in segment \(i\) defined as:

\[
av_i = \frac{value_i(n_i)}{n_i}.
\]

\(^7\)The sum of normally distributed random variables which are not independent is not necessarily a normally distributed random variable.
The average profit per customer depends therefore only on the type of segment and not on the number of customers in the segment. In this way one can decouple the decision variables and the correlation. Considering that by definition of correlation

\[ \text{cov}(\text{value}_i(n_i), \text{value}_j(n_j)) = \sigma_i \sigma_j \rho_{i,j}, \]

the variance of the customer equity becomes:

\[ \sigma_{CE}^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{i,j} \sqrt{n_i^3} \sqrt{n_j^3}. \] (4.22)

The covariance between the average profits per customer can be calculated if there is some common event related to two segments. For instance, correlation can be measured by considering the time moment as the linking event between two segments. In this case, the average profits per customer for two segments belonging to two different geographical regions can be compared each month and the correlation can easily be computed, without considering the number of people in each segment. In this way seasonal correlations between regions can be detected. For instance, the summer season in Italy corresponds to the winter season in New Zealand and a company having marketing activities in both regions can take advantage of this (probably negative) correlation to reduce (by diversification) the risk of the overall return on investment.

The optimization problem in the case of normal distribution approximation and correlations between the value of different segments is therefore:

\[
\min_{n_1, \cdots, n_N} \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{i,j} \sqrt{n_i^3} \sqrt{n_j^3} \\
\sum_{i=1}^{N} n_i \mu_i = \hat{\mu}_{CE} \\
\sum_{i=1}^{N} n_i \mu_{c,i} \leq \beta \\
0 \leq n_i \leq \max(i) \quad i : 1, \cdots, N. \]

(4.23)

4.6 Numeric examples

In this section I provide an example of the optimization of a customer portfolio. The example that I discuss is inspired from a real case involving a major European airline. For confidentiality reasons and for the sake of simplicity I do not make use of any real data, but use fictitious figures.
In this example, an airline has four main customer segments: platinum, gold, silver, base. This segmentation is based on the loyalty of the customer base. Loyalty is interpreted as the intensity of the use of the company’s services and is measured in terms of points collected in some specified time interval. Customers acquire points when they buy an airline ticket. According to this segmentation the platinum customers are those who acquired a very large number of points, followed by the gold and the silver customers. Customers that did not gather enough points are defined as base customers. They are characterized by a sporadic behavior and constitute the majority of the customers. The following table describes the characteristics of each segment in terms of number of people in the segments, profits and costs distributions.

<table>
<thead>
<tr>
<th>Segment</th>
<th>Platinum</th>
<th>Gold</th>
<th>Silver</th>
<th>Base</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean profit (euro)</td>
<td>14000</td>
<td>8000</td>
<td>4000</td>
<td>2000</td>
</tr>
<tr>
<td>standard deviation profit (euro)</td>
<td>7000</td>
<td>4000</td>
<td>1500</td>
<td>700</td>
</tr>
<tr>
<td>mean cost (euro)</td>
<td>500</td>
<td>300</td>
<td>180</td>
<td>100</td>
</tr>
<tr>
<td>number of people</td>
<td>1500</td>
<td>3000</td>
<td>6000</td>
<td>70000</td>
</tr>
</tbody>
</table>

Table 4.1: Customer segments (first example).

I use the optimization problem (4.20) to find the optimal customer portfolio. Therefore it is not necessary to specify the standard deviation of the cost per individual for each segment. The profits and the costs refer to one year. The costs represent direct marketing costs such as campaigns, points\(^8\) and the use of the lounge in the case of platinum, gold, and silver customers. From table 4.1, one notes that the platinum customers generated on average the most value in one year (i.e. 14000 euros) yet the uncertainty measured by the standard deviation is very high (i.e. 7000 euro). Therefore these customers represent a high-value but also a risky segment. On the other hand, base customers generate much less profit over one year but their standard deviation is much smaller.

In this example I consider a marketing budget of 250000 euros. Costs are assumed to be directly linked to the generated revenues. The portfolio of customers minimizing the risk for a given expected value is the solution of the following optimization problem:

\[
\begin{align*}
\min_{n_1, \ldots, n_4} & \quad n_1 \cdot (7000)^2 + n_2 \cdot (4000)^2 + n_3 \cdot (1500)^2 + n_4 \cdot (700)^2 \\
& \quad n_1 \cdot (14000) + n_2 \cdot (8000) + n_3 \cdot (4000) + n_4 \cdot (2000) = \mu_{CE} \\
& \quad n_1 \cdot (500) + n_2 \cdot (300) + n_3 \cdot (180) + n_4 \cdot (100) \leq 250000 \\
& \quad 0 \leq n_1 \leq 1500, \quad 0 \leq n_2 \leq 3000 \\
& \quad 0 \leq n_3 \leq 6000, \quad 0 \leq n_4 \leq 70000.
\end{align*}
\]

\(^8\)Each point has a value for the company since it can be used to buy flight tickets.
4.6. Numeric examples

By varying the parameter $\hat{\mu}_{CE}$ one obtains the efficient frontier which is shown in figure 4.1. Note that if the expected customer equity increases, the risk measured by the standard deviation of the portfolio increases as well. The efficient frontier is computed by fixing the expected customer equity and by imposing that the marketing investment is less or equal than 250000 euros. Each point $(\mu, \sigma)$ lying on the efficient frontier, represents the portfolio with expected value $\mu$ which has the minimum possible risk $\sigma$.

The composition of the optimal customer portfolio is illustrated in figure 4.2. For each segment, the number of customers to target is shown as a function of the expected customer equity. For low values, the optimal customer portfolio consists only of base customers. This is because given the marketing budget, high-value customers such as platinum or gold are not needed in order to achieve the predefined level of expected customer equity. For higher levels of expected value, the budget constraint does not allow us to target many base customers. Segments with a higher return on investment but with higher volatility as well need to be targeted in this case.

Figure 4.3 shows that the total number of customers in the portfolio decreases when the customer equity increases. In fact fewer customers are targeted in order to achieve a high return on investment. From the same figure one can note that the marketing costs increase.
linearly until the maximum available budget is reached. After that point, the costs remain constant but the expected customer equity increases at the price of an increase in risk. The value-risk tradeoff is illustrated in the bottom chart on the right, where the risk is plotted against the expected return on investment. The higher the risk, the higher is the expected return on investment in the case of efficient portfolios.

The efficient frontier in figure 4.1 shows all the feasible optimal customer portfolios. The final portfolio choice depends on the risk degree that the decision maker (i.e. the company) is willing to accept. The risk increases with the expected value of the portfolio, however, in this case, the expected value increases faster than the risk. This fact may not be easily visible from the above chart because the two axis have a different scale. The Sharp ratio $\xi$ is defined as the ratio between the expected value and the variance of an investment, i.e.

$$\xi = \frac{\mu}{\sigma}.$$  

This ratio tells us how much expected value is generated per unit of risk, where the risk is measured using the standard deviation. The Sharp ratio can be used to compare different investments having different value and risk tradeoffs. In fact, a portfolio with a high standard deviation can be preferred over a portfolio with a low risk if the expected value of the risky
Figure 4.3: Marketing investment and ROI (first example).

portfolio is high.

Figure 4.4 plots the Sharp ratio against the marketing investment. Note that the ratio increases non-linearly with the marketing investment, this means that portfolios obtained by investing the entire budget (i.e. 250000 euros) exhibit the highest return per unit of risk. The above figure shows that when the investment reaches the available budget, different portfolios with different Sharp ratios can be generated. The portfolio with the highest Sharp ratio leads to the highest expected customer equity (or highest expected return on investment). This portfolio exhibits the highest risk, as shown in figure 4.1. However, the expected return per unit of risk is the highest (the Sharp ratio is equal to 143). Figure 4.5 shows the distributions of three portfolios in the efficient frontier for the case where the entire marketing budget is invested. The three portfolios lead to different expected returns on investment because their expected values differ. For each distribution, the maximum and the minimum outcomes are specified. I define the maximum outcome as having value $\mu + 2\sigma$ and the minimum outcome as having value $\mu - 2\sigma$. For normal distributions, 95.5% of the outcomes fall into this interval. Note that the worst outcome of the portfolio with the highest expected value is only slightly less than the best outcome of the portfolio with the second highest expected value. The distribution on the left, corresponding to an expected value of 6 million, has the smaller
variance (i.e. risk) but its best value is smaller than the worst value of the distribution in the middle. In practice, the distribution with the lowest variance should never be selected. A value-risk tradeoff does not take place in this example. In fact, the portfolio with the highest expected value and the highest variance would probably be selected by any rational decision maker because the increment in value compensates for the increment in risk.

Taking a new example, two different marketing campaigns are designed for two different geographic regions: Europe and USA. Based on historical data, profit and cost distributions can be estimated. The input data for the model is summarized in table 4.2. The high standard deviation might seem unrealistic if compared with the expected value. In this example, I focus on the mathematical optimization problem and on the resulting portfolios rather than on the marketing scenario.

<table>
<thead>
<tr>
<th>Segment</th>
<th>Europe</th>
<th>USA</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean profit (euro)</td>
<td>633.33</td>
<td>232.14</td>
</tr>
<tr>
<td>standard deviation profit (euro)</td>
<td>2742.4</td>
<td>316.93</td>
</tr>
<tr>
<td>mean cost (euro)</td>
<td>400</td>
<td>150</td>
</tr>
<tr>
<td>number of people</td>
<td>300</td>
<td>900</td>
</tr>
</tbody>
</table>

Table 4.2: Customer segments (second example).
The optimal portfolios are found by solving the following optimization problem:

\[
\begin{align*}
\min_{n_1, n_2} & \quad n_1 \cdot (2742.4)^2 + n_2 \cdot (316.93)^2 \\
& \quad n_1 \cdot (633.33) + n_2 \cdot (232.14) = \mu_{CE} \\
& \quad n_1 \cdot (400) + n_2 \cdot (150) \leq 204000 \\
& \quad 0 \leq n_1 \leq 300, \quad 0 \leq n_2 \leq 900.
\end{align*}
\]

(4.25)

The efficient frontier, the Sharp ratio, and the number of customers of each segment are represented in figure 4.6.

In this case, the Sharp ratio is not monotonically increasing. There is a portfolio obtained without allocating all the marketing budget, which has the highest expected value per unit of risk. If one compares the efficient frontier in figure 4.6 with that in figure 4.1, one notes that the risk is increasing much more rapidly in this example and that the standard deviation is an order of magnitude smaller than the expected value. In the first example the standard deviation is two orders of magnitude smaller than the expected value. This can be seen also by comparing the Sharp ratios, the portfolio with the maximum expected value has a
ratio of 143 in the first example and 6.72 in the second example. This suggests that in the second example there is an effective tradeoff between value and risk. In order to evaluate this tradeoff I compare the distributions of three efficient portfolios in figure 4.7. The portfolio with the lowest expected value is obtained by investing only part of the available marketing budget. The other two portfolios are obtained investing the entire marketing budget.

The portfolio with the highest expected value (P1) has the highest standard deviation; the worst outcome of this portfolio is smaller than the worst outcomes of the two other portfolios. Risk-averse decision makers would prefer the portfolio with the second highest expected value (P2), since this is the portfolio with the best worst case. If one compares the portfolio with the second highest expected value (P2) with the portfolio with the lowest expected value (P3), one concludes that there is no value-risk tradeoff. In fact, the outcomes of P2 are always preferred over the outcomes of P3.
4.7 Parameter uncertainty

Until now I assumed that the forecasted profit distributions and in particular the mean and the variance of the various segments were known. In practice, these quantities need to be estimated from the available data. The uncertainties resulting from the estimation can be incorporated into the portfolio risk. In general, there are three sources of uncertainties:

- model uncertainty due to the uncertainty in the model used to compute the profit distributions in a given period,
- parameter uncertainty due to the uncertainties in the parameters of the employed model,
- stochastic uncertainty due to the probabilistic nature of the process.

I do not deal with the model risk, since this information is usually not provided. Rather, I discuss how the risk due to the uncertainty in the parameters of the model and by the stochastic nature of the possible outcomes affects the final portfolio risk. As discussed previously, the
mean of the customer equity portfolio is more sensitive to changes in the decision variables than the variance. Therefore I consider only the uncertainty in the mean of the segment distribution and neglect the error on the standard deviation. This approximation is also motivated by simplicity reasons. In fact, the normal distribution assumptions holds if one considers the uncertainties in the mean and assumes a given prior distribution of the mean profit of each segment. According to [19], I assume that the prior distribution of the mean is normally distributed. In general, if one knows the conditional probability distribution \( f_{p|\theta} \) of a random variable \( p \) given a parameter \( \theta \in \Theta \) and the probability distribution \( f_\theta \) of the parameter \( \theta \), one can derive the unconditional probability distribution \( f_p \) of \( p \) as follows [60]:

\[
\int f_{p|\theta} f_\theta d\theta.
\]

In the case that the parameter \( \theta \) is normally distributed and \( f_{p|\theta} \) is normally distributed the following property holds.

**Property 4.7.1.** Given a normally distributed random variable with mean \( \mu \) and variance \( \sigma^2 \)

\[
p_\mu \sim \mathcal{N}(\mu, \sigma^2),
\]

where the mean \( \mu \) is in turn normally distributed

\[
\mu \sim \mathcal{N}(m, \nu^2),
\]

then the unconditional distribution of the random variable \( p \) is

\[
p \sim \mathcal{N}(m, \sigma^2 + \nu^2).
\]

**Proof.** We define \( z \) and \( y \) as two independent random variables normally distributed with zero mean and unitary variance. The quantity \( p_\mu \) is equal to

\[
p_\mu = \mu + \sigma z.
\]

Since \( \mu \) is equal to

\[
\mu = m + \nu y,
\]

the random variable \( p \) can be obtained by substituting \( \mu \) in the equation of \( p_\mu \)

\[
p = m + \nu y + \sigma z \sim \mathcal{N}(m, \nu^2 + \sigma^2).
\]

The value distribution of a given segment \( i \), having mean \( \mu_i \) and \( n_i \) targets is

\[
value_{i|\mu_i}(n_i) \sim \mathcal{N}(n_i\mu_i, n_i\sigma_i^2),
\]
4.7. Parameter uncertainty

If the mean $\mu_i$ is normally distributed, i.e.

$$\mu_i \sim \mathcal{N}(m_i, \nu_i^2),$$

the total mean of the segment obtained by targeting all the people is distributed as follows

$$n_i \mu_i \sim \mathcal{N}(n_i m_i, n_i^2 \nu_i^2).$$

According to property 4.7.1, the unconditional distribution of the value incorporating the uncertainty about the mean is therefore

$$\text{value}_i(n_i) \sim \mathcal{N}(n_i m_i, n_i \sigma_i^2 + n_i^2 \nu_i^2).$$

Here I suppose that the variance $\sigma_i^2$ of segment $i$ is known. Finally, the customer equity is distributed as follows

$$CE(n_1, \ldots, n_N) \sim \mathcal{N}\left(\sum_{i=1}^{N} n_i m_i, \sum_{i=1}^{N} n_i \sigma_i^2 + \sum_{i=1}^{N} n_i^2 \nu_i^2\right).$$

The optimal customer portfolio is therefore the solution of the following optimization problem:

$$\min_{n_1, \ldots, n_N} \sum_{i=1}^{N} n_i \sigma_i^2 + \sum_{i=1}^{N} n_i^2 \nu_i^2$$

$$\sum_{i=1}^{N} n_i m_i = \hat{\mu}_{CE}$$

$$\sum_{i=1}^{N} n_i \mu_{c,i} \leq \beta$$

$$0 \leq n_i \leq \max(i) \quad i : 1, \ldots, N,$$

where the mean costs per segment $\mu_{c,i}$ are assumed to be known, in fact both the parameter uncertainty and the stochastic risk can be transferred to the profit distribution which depends on the cost and the revenue of each segment. Problem (4.26) is a quadratic optimization problem, it can be solved exactly using many commercial solvers.

Next I use a simple model based on the historical data to estimate the mean and the variance of the future profit for each segment and I discuss the uncertainty about the mean.

Under the assumption that the profit distributions are stationary and that historical data is available, the empirical mean and the empirical variance can be computed for each segment. This assumption makes sense if the marketing actions and the customer base do not change over time. For the Central Limit Theorem, the empirical mean $\overline{x}_n$ follows a normal distribution if the size $n$ of the samples is large, i.e.

$$\lim_{n \to \infty} \overline{x}_n = \lim_{n \to \infty} \frac{\sum_{i=1}^{n} x_i}{n} \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right),$$
where \( x_i \) are identically distributed and independent samples with mean \( \mu \) and variance \( \sigma^2 \). If the standard deviation is not known, the empirical variance can be used, then the distribution of the empirical mean is a student distribution with \( n - 1 \) degrees of freedom. However, if the number of samples is large, the student distribution tends to a normal distribution [60]. Therefore the empirical variance \( s^2 \) can be used:

\[
s^2_n = \frac{\sum_{i=1}^{n}(x_i - \bar{x}_n)^2}{n - 1}.
\]

Next I analyze the consequence of the uncertainty on the mean in the distribution of a segment. Given \( k \) samples, the empirical mean is distributed as a normal random variable. Given the empirical mean, the true mean can be modeled as a normal distribution. In fact,

\[
x_k \sim \mathcal{N}(\mu, \sigma^2_k) \rightarrow \bar{x}_k = \mu + \frac{\sigma}{\sqrt{k}}z \rightarrow \mu = \bar{x}_k - \frac{\sigma}{\sqrt{k}}z \sim \mathcal{N}(\bar{x}_k, \frac{\sigma^2}{k}),
\]

where \( z \) is normally distributed with mean zero and unitary variance.

According to this model

\[
\nu_i^2 = \frac{\sigma_i^2}{k_i},
\]

where \( k_i \) is the number of samples used to estimate the mean of segment \( i \). The value of the customer equity is therefore distributed as follows:

\[
CE(n_1, \cdots, n_N) \sim \mathcal{N}\left(\sum_{i=1}^{N} n_i m_i, \sum_{i=1}^{N} n_i \sigma_i^2 + \sum_{i=1}^{N} n_i^2 \frac{\sigma_i^2}{k_i}\right).
\]

The uncertainty on the mean has a relevant effect if the empirical mean has been calculated using less customers than the ones that are targeted. In fact, if \( k_i \geq n_i \) then

\[
\sum_{i=1}^{N} n_i^2 \frac{\sigma_i^2}{k_i} = \sum_{i=1}^{N} n_i \alpha_i \sigma_i^2,
\]

where \( \alpha_i = n_i(k_i)^{-1} \leq 1 \). In this case the variance of the customer equity depends linearly on the decision variables \( n_i \):

\[
\sigma_{CE}^2 = \sum_{i=1}^{N} n_i \sigma_i^2 + \sum_{i=1}^{N} n_i^2 \frac{\sigma_i^2}{k_i} = \sum_{i=1}^{N} n_i \sigma_i^2 + \sum_{i=1}^{N} n_i \alpha_i \sigma_i^2 = \sum_{i=1}^{N} n_i \sigma_i^2 (1 + \alpha_i).
\]

In fact, \( \alpha_i \) is bounded for every value of \( n_i \). On the other hand, if \( n_i > k_i \), the variance depends quadratically on the decision variables and for large values of \( n_i \) the risk due to the uncertainty on the mean of the segments can penalize high-value segments with high variance and not well known mean. To show this, I consider again the example described in table 4.1, but I assume that the empirical means of each segment have been calculated using
### 4.7. Parameter uncertainty

<table>
<thead>
<tr>
<th>Segment</th>
<th>Platinum</th>
<th>Gold</th>
<th>Silver</th>
<th>Base</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of samples</td>
<td>50</td>
<td>100</td>
<td>200</td>
<td>1000</td>
</tr>
</tbody>
</table>

Table 4.3: Sample size of the customer segments.

a limited sample. Table 4.3 describes how many customers of each segment have been used in order to compute the mean profit per customer.

By solving optimization problem (4.26) with \( \nu_i^2 = \sigma_i^2 (k_i)^{-1} \) one can find the efficient frontier. Figure 4.8 shows the distribution of three portfolios in the efficient frontier. By comparing these distributions with the case in which the uncertainty in the parameters is not considered (i.e. figure 4.5), one can note that the tradeoff between value and risk is more evident now. The worst case of the portfolio with the highest expected value (P1) is smaller than the worst case of the second highest portfolio (P2). This is not the case when the risk due to the sample mean is not taken into account. Segments with high-value customers represent usually the smaller segments, for such segments the estimation of the mean can be very imprecise due to the small number of samples.

![Figure 4.8: Distributions of three efficient portfolios with parameter uncertainty.](image-url)
In a marketing context it is realistic to assume that the sample on which some statistics (e.g. expected return, expected response rate, etc) are estimated is much smaller than the potential targets. In fact, in the case that new campaigns or products are designed, historical data is typically not available. In these cases the usual approach is to test the new marketing activities on a limited sample of customers and then select those activities with the best performance. Consequently, these activities are applied to a much larger set of customers than those considered in the original sample.

As the parameter risk depends quadratically on the number of targeted customers, if the marketing budget is incremented and more people are targeted, the value-risk tradeoff will be accentuated. The Sharp ratio is no longer as large as in the case when parameter uncertainty is not considered. However, as in the case of table 4.3, the Sharp ratio increases with the number of targeted customers and becomes very large if many customers are targeted. The value-risk tradeoff is not very relevant and the customer portfolio with the highest expected value (and the highest risk) might still be the best choice for any rational decision maker. In fact, the difference between the worst cases of portfolios P1 and P2 might not be significative enough to justify a risk-averse preference for portfolio P2. These conclusions apply to the particular example and are not generalizable to other cases.

### 4.8 The value-risk tradeoff

As discussed previously, portfolios with a high risk can be preferred over portfolios with lower risk and lower expected value. In fact, the worst case of a high-risk portfolio might be better than the best case of a low-risk portfolio.

In the case of normal distributions, a confidence interval in which $95.5\%$ of the values fall is described by

$$[\mu - 2\sigma, \mu + 2\sigma],$$

where $\mu$ and $\sigma$ are, respectively, the mean and the standard deviation. The upper and lower bound defining the interval are equal to

$$\mu \pm 2\sigma = \mu \left(1 \pm \sqrt{2} \sigma \mu \right).$$

By introducing the Sharp ratio $\xi$ the interval becomes

$$\left[\mu \left(1 - \frac{2}{\xi}\right), \mu \left(1 + \frac{2}{\xi}\right)\right].$$

If $\xi$ is high (e.g. 143) the confidence interval is relatively close to the mean $\mu$. In fact, the relative deviance $r$ from the mean is given by

$$r = \frac{2}{\xi},$$
4.8. The value-risk tradeoff

and is very small for high values of the Sharp ratio. Using the relative deviance \( r \) the confidence interval can be written as

\[
[\mu - \mu r, \mu + \mu r].
\]

If \( r \) is very small, the main criteria to evaluate a portfolio is its expected value since most of the random outcomes will be relatively close to the mean.

As to the customer equity, if the parameter risk is not considered the Sharp ratio of a given segment \( i \) is defined by

\[
\xi_i = \frac{\mu_i n_i}{\sigma_i \sqrt{n_i}} = \frac{\mu_i}{\sigma_i} \sqrt{n_i}.
\]

Therefore, if a large number of customers is targeted

\[
\lim_{n_i \to \infty} \xi_i = \infty, \quad \lim_{n_i \to \infty} r_i = 0.
\]

For large values of \( n_i \) the confidence interval will be small, and the risk that the outcomes will be distant from the mean is reduced. This effect is due to the assumption that customers are independent, therefore when customers are aggregated into a segment, the risk is automatically reduced through diversification (see chapter A).

For instance, if there are two segments with expected values \( \mu_1 \) and \( \mu_2 \), the two confidence intervals as defined previously could not have any point in common if \( \mu_1 \gg \mu_2 \), in this case the optimal portfolio would be given by maximizing the expected value without taking risk into consideration.

The above theoretical aspects are confirmed by the efficient frontier in figure 4.1 and the Sharp ratio in figure 4.4. These charts show that if the expected customer equity is maximized by targeting many customers belonging to one or more segments and by exhausting all the available marketing budget, the standard deviation increases but the relative risk decreases. In other words, if one takes the portfolio with the maximum expected value, even if the variance is maximized, the worst case is still better than the best case of a portfolio with lower expected value and lower risk.

By considering the parameter uncertainty, the limit of the Sharp ratio of segment \( i \) is:

\[
\lim_{n_i \to \infty} \xi_i = \frac{\mu_i n_i}{\sqrt{n_i \sigma_i^2 + n_i \nu_i^2}} = \frac{\mu_i}{\sigma_i \sqrt{n_i}} = \frac{\mu_i}{\nu_i} < \infty.
\]

Therefore the value-risk tradeoff becomes more relevant if the parameter uncertainty is considered. In fact, the Sharp ratio is limited in this case.

The portfolio which maximizes the expected value without taking risk into consideration can easily be found by allocating resources to the customer segment starting with the highest expected return on investment and continuing with the other segments until the marketing budget is exhausted. Using this algorithm one would obtain exactly the same solution as the
one found by solving the linear optimization problem (4.20) and by imposing the maximum expected customer equity. In fact, the first segment to target is the platinum which has the highest expected return on investment; the marketing budget is exhausted by targeting 500 customers in this segment.

### 4.9 Conditions for risk management

There is not always a value-risk tradeoff justifying a risk-sensitive allocation of resources. In this section I establish some sufficient and necessary conditions justifying the application of risk management in the case of customer portfolios. I start by considering the case of only two segments and afterwards generalize the results to the case of many segments. The basic principle justifying any risk management approach is the following.

**Principle.** *There is a value-risk tradeoff if and only if two portfolios, P1 and P2, can be created with the following properties. The expected value of P1 is higher than the expected value of portfolio P2. However, the worst outcome of portfolio P1 is worse than the worst outcome of P2.*

In the case of random variables the worst case can be defined as a percentile. For normal distributions, the probability that an outcome will be less or equal to the mean minus two times the standard deviation is less or equal than 2.25%, i.e.

$$P(x \leq \mu - 2\sigma) \leq 2.25\%.$$  

Therefore I define the worst possible case \(x_-\) as

$$x_- = \mu - 2\sigma.$$

According to the above principle, there is a value-risk tradeoff between portfolio P1 and portfolio P2 if and only if the following conditions are verified:

$$\mu_{P1} > \mu_{P2}$$
$$\mu_{P1} - 2\sigma_{P1} < \mu_{P2} - 2\sigma_{P2}. \quad (4.27)$$

Let us define \(n_{i,P1}\) and \(n_{i,P2}\) as the number of customers in segment \(i\) targeted in portfolio P1 and P2 respectively. In the case of two segments, if parameter uncertainty is not considered, the above conditions are equivalent to the following:

$$\mu_1(n_{1,P1} - n_{1,P2}) + \mu_2(n_{2,P1} - n_{2,P2}) > 0$$
$$\mu_1(n_{1,P1} - n_{1,P2}) + \mu_2(n_{2,P1} - n_{2,P2}) <$$
$$2 \left( \sqrt{n_{1,P1}\sigma_1^2 + n_{2,P1}\sigma_2^2} - \sqrt{n_{1,P2}\sigma_1^2 + n_{2,P2}\sigma_2^2} \right). \quad (4.28)$$
In the general case of $N$ segments and including parameter risk, the conditions (4.27) are equivalent to the following:

$$\sum_{i=1}^{N} \mu_i (n_{i,P_1} - n_{i,P_2}) > 0$$

$$\sum_{i=1}^{N} \mu_i (n_{i,P_1} - n_{i,P_2}) < \left( \frac{\sum_{i=1}^{N} n_{i,P_1} (\sigma_i^2 + n_{i,P_1} \nu_i^2)}{\sqrt{N}} - \frac{\sum_{i=1}^{N} n_{i,P_2} (\sigma_i^2 + n_{i,P_2} \nu_i^2)}{\sqrt{N}} \right).$$

The above conditions depend on various parameters, in general the quantities $\mu_i$ and $\sigma_i$ of the different segments are assumed to be known. The number of targets are constrained by the available marketing budget and by the maximum number of customers in each segment. In practice, these conditions are not very helpful in finding two portfolios with a value-risk tradeoff. In fact, it is difficult to solve them with respect to the decision variables $n_{i,P_1}$ and $n_{i,P_2}$ given the mean and variance of the different segments and the constraints on the number of customers. Rather, they can be used to check if there is a value-risk tradeoff between two known portfolios.

From equation (4.28) one can derive a sufficient (but not necessary) condition for the value-risk tradeoff between two portfolios. By imposing $n_{1,P_2} = n_{2,P_1} = 0$ and $n_{1,P_1} = n_{2,P_2} = n$ one obtains the condition:

$$\mu_1 - \mu_2 > 0$$

$$\sqrt{n}(\mu_1 - \mu_2) < 2(\sigma_1 - \sigma_2).$$

This condition involves less parameters and is therefore easier to check than the more general case. If the condition is true, then two portfolios can be constructed which are not diversified and which target the same number of customers and which exhibit a value-risk tradeoff. In this case diversification might help to reduce the risk.

For example, I consider the two customer segments defined in table 4.4. If the available marketing budget is 220000, then two portfolios which exhaust all the budget can be built. These portfolios are defined in table 4.5. Using equations (4.27), without considering the

<table>
<thead>
<tr>
<th>Segment</th>
<th>$\mu$</th>
<th>$\sigma$</th>
<th>cost</th>
<th>max</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>300</td>
<td>100</td>
<td>150</td>
<td>1000</td>
</tr>
<tr>
<td>B</td>
<td>200</td>
<td>50</td>
<td>100</td>
<td>2045</td>
</tr>
</tbody>
</table>

Table 4.4: Segments A and B.
Parameter uncertainty, one can verify that P1 and P2 have a value-risk tradeoff, in fact:

\[ \mu_{P1} - \mu_{P2} = 100 > 0 \]

\[ \mu_{P1} - \mu_{P2} - 2(\sigma_{P1} - \sigma_{P2}) = -1798.8 < 0. \] (4.31)

Assuming that the total value of the two portfolios is normally distributed, then the portfolios have the distributions shown in figure 4.9.

The portfolio with minimum risk for an expected value of 440000 is given by portfolio P2 according to the solution of problem (4.20). This is because portfolio P2 almost reaches the expected value of 440000 but has much less variance than portfolio P1. As it is not possible to target a fractional number of customers, portfolio P2 is the efficient portfolio which more closely approaches a portfolio with expected value of 440000 and minimum risk.
According to the standard Markowitz portfolio optimization theory, portfolio P1 would never be selected since it is not part of the efficient frontier. However, some decision makers might prefer P1 over P2 because the additional risk of P1 (measured by the standard deviation) might cause an outcome which is larger than P2. Such a decision maker is risk seeking. In order to optimally build customer portfolios it is necessary to model this kind of preference. Expected utility theory provides a framework to build portfolios which are consistent with the risk attitude of the decision maker. Moreover, utility theory provides means to analytically estimate the risk attitude of rational decision makers (see chapter A).

4.10 Solving the value-risk tradeoff

The Markowitz framework does not specify exactly which portfolio on the efficient frontier should be selected, moreover, the investor is assumed to be always risk averse. Using utility theory one can generalize the customer portfolio selection problem to include the subjective risk attitude of the decision maker. This allows us to localize the position of the ideal portfolio in the efficient frontier and thus to solve the value-risk tradeoff. The price to pay for using utility theory is the requirement that the decision maker’s utility function must be derived. This task can be rather complex in practice [43]. According to theorem A.3.3, if an investor has constant (positive or negative) local risk aversion, then his utility function must be of the form:

\[ u(x) = a - be^{-cx}, \]

where \( b > 0 \) and \( c > 0 \) in the case of risk aversion and \( c < 0 \) in the case of risk proneness. As any linear combination of this utility function leads to the same ranking of preferences (theorem A.3.2), I consider \( a = 0 \) and \( b = 1 \) for risk aversion and \( b = -1 \) for risk proneness.

According to section 4.3, I assume that the customer equity is normally distributed with mean \( \mu_{CE} \) and variance \( \sigma_{CE}^2 \). Under the assumption that investors have constant local risk aversion, the utility is exponential. In this case the expected utility can easily be calculated, in fact the exponential function of a normally distributed random variable, i.e. the customer equity, is distributed like a lognormal (LN) random variable with the following mean and variance [60]:

\[ e^{-c \cdot CE} \sim LN \left( e^{-c \mu_{CE} + \frac{c^2}{2} \sigma_{CE}^2}, e^{-2c(\mu_{CE} - c \sigma_{CE}^2) - e^{-c(2\mu_{CE} - c \sigma_{CE}^2)}} \right). \]

In the case of risk aversion the expected utility is:

\[ \mathbb{E}[u(CE)] = \mathbb{E}[-e^{-c \cdot CE}] = -\mathbb{E}[e^{-c \cdot CE}] = -e^{-c \mu_{CE} + \frac{c^2}{2} \sigma_{CE}^2}. \]

The expected utility is maximized if the exponent is minimized, i.e.

\[ \max \ \mathbb{E}[u(CE)] \to \min \ -c \cdot (\mu_{CE} - \frac{c}{2} \sigma_{CE}^2). \]
since \( c > 0 \) the above problem becomes

\[
\max \, \mu_{CE} - \frac{c}{2} \sigma_{CE}^2,
\]

this optimization problem corresponds to the Lagrangian Markowitz form (A.7) on page 125. The risk-aversion coefficient \( c \) is known if the utility function of the decision maker is known. The efficient frontier is obtained by varying the risk-aversion coefficient, therefore the parameter \( c \) identifies a particular optimal portfolio in the efficient frontier. If the decision maker is risk prone, i.e. \( c < 0 \) and \( b = -1 \), then the same type of optimization problem is obtained:

\[
\max \, \mathbb{E}[u(CE)] \rightarrow \max \, \mu_{CE} - \frac{c}{2} \sigma_{CE}^2.
\]

The risk-aversion coefficient \( c \) specifies the tradeoff between expected value (\( \mu_{CE} \)) and risk (\( \sigma_{CE} \)) and can be estimated using the techniques described in section A.3.3. Note that a risk-prone investor does not penalize the variance. In the case of two portfolios with the same expected value, a risk-prone investor would select the portfolio with the larger variance. For example, in figure 4.9, a risk-averse decision maker would select portfolio P2 while a risk-prone investor would select portfolio P1.

## 4.11 Conclusions and managerial implications

In this chapter I describe how customer portfolios can be managed using financial optimization techniques to address the value-risk tradeoff. The nature of the value-risk tradeoff does depend on the particular case, i.e. the customer value distributions. Unlike assets exchanged in financial markets, when evaluating the risk profile of customer assets there is not always an effective value-risk tradeoff. In fact, as shown using some examples, a risk-averse decision maker might prefer a high-value and high-risk customer segment over a customer segment with lower expected value and lower risk. This is the case when the value distributions of the two segments do not overlap and the worst case of the risky segment is still better than the best case of the less risky segment. Financial assets are usually exchanged on a nearly efficient market, therefore the value-risk tradeoff is forced by the laws of demand and supply. As this mechanism does not apply for customer assets, the value-risk tradeoff is not always relevant when building customer portfolios.

By analyzing the Sharp ratio of customer portfolios, I discuss the conditions under which a value-risk tradeoff is a relevant issue. For instance, the uncertainty due to the model parameters can have a large impact on the tradeoff as shown by comparing figures 4.8 and 4.5. The uncertainty in the model parameters depends on the reliability of the estimated parameters, which is sensitive to the quantity of data used in the estimation process.
4.11. Conclusions and managerial implications

As in a marketing context it is realistic to assume that the sample on which some statistics (e.g. expected return, expected response rate, etc.) are estimated is much smaller than the potential targets, the consideration of the risk embedded in the predictive model is relevant when planning marketing campaigns and allocating marketing resources. In fact, when new campaigns or products are designed historical data is typically not available. The usual approach is to test the new marketing activities on a limited sample of customers and select those activities with the best performance and consequently apply them in the future to a much larger set of customers than those considered in the sample.

Finally, I describe how utility theory can be used to build customer portfolios which are consistent with the risk attitude of the decision maker.
Chapter 5

Predicting individual value distributions

5.1 Introduction

In chapter 4 I assumed that the profit (i.e. value) distributions of the customers were known; moreover I assumed that customers belonging to the same segment were characterized by the same distribution. This assumption implies that customers in one segment are considered as similar and that their profit distribution can be estimated by sampling independently from a given segment. As a consequence, the distribution of a given customer depends on the particular segmentation criterion. As all the customers belonging to a given segment follow the same profit distribution, it is possible to associate, to a given customer, an interval of possible profits with their realization probabilities even if only one, or few, transactions (i.e. profit outcomes) of that customer have been observed. Therefore, I am implicitly borrowing strength from the database to estimate the customer’s profit distribution.

In this chapter I address the problems of predicting the future value distribution of individual customers and of allocating a limited marketing budget in order to build a customer portfolio that maximizes the expected future value while taking into account the uncertainty of that value. In most of the current literature, the expected customer lifetime value is estimated (e.g. [8], [40], [11], [34], [33]) without considering heterogeneity across customers. More sophisticated models consider customer heterogeneity (e.g. [78], [74]) by modeling the individual migration or survival probabilities. However, all the current models focus on the expected customer lifetime value.

My approach enables to estimate the probability distribution of the future lifetime value taking into account heterogeneity across customers, in this way customers can be characterized and segmented not only according to the future expected value but also according to the relationship risk. Risk-averse marketing managers would probably prefer a portfolio of stable customers for which there is not much uncertainty concerning their future value.
On the other hand, marketing managers acting in circumstances of market expansion might prefer a portfolio of potentially high-value but as well high-risk customers. The risk can be due to the fact that not much historical data is available to estimate the future lifetime value (e.g. because of entering a new market), or to the fact that customers behave very randomly and a correlation between observed features (e.g. age, frequency, recency, etc.) and lifetime value is very weak.

As marketing data is often sparse at the individual customer level [76], I use mixtures of Gaussians [86] in order to borrow strength from the data and estimate the individual lifetime value probability densities. Once the future densities are known, utility theory [62], [70] can be used to allocate marketing resources rationally by taking into account the risk due to the intrinsic stochastic nature of the future predictions. Marketing managers can therefore allocate their marketing budget so as to meet financial criteria, based on the maximization of the expected value and the minimization of the risk [27].

The chapter is organized as follows. After an introduction of the theory of Gaussian mixtures and the estimation of the model parameters (sections 5.2 and 5.3), I discuss how to estimate conditional distributions (section 5.4). In section 5.5 the individual customer lifetime value distributions are estimated using the Gaussian mixture.

In section 5.6 I introduce the principle of investment selection using expected utility and derive a closed form to predict the expected future utility at an individual customer level. This allows us to build customer portfolios by selecting the customers according to their predicted expected utility. The developed approach extends the standard practice because the uncertainty about the future prediction is explicitly taken into account by using utility theory. Moreover, the approach generalizes the standard practice because in the case of risk neutrality of the decision maker, the expected utility is equal to the expected value.

I describe a case study in section 5.7 showing several marketing applications, including the forecasting of future value, the identification of defectors, and the establishment of a customer portfolio. Particular attention is paid to the validation of the mixture model. The performance is compared with that of other well-known state-of-the-art predictive algorithms. Using bootstrap [26] to estimate the value distribution of the resulting portfolios, I show that a risk-sensitive resource allocation leads to customer portfolios with a lower downside risk.

Finally, in the conclusion (section 5.8) I summarize the main findings and discuss the managerial implications.
5.2 Gaussian mixtures

There is a long tradition of using mixtures of distributions in modern statistics [86]. A probability density function of a mixture of \( m \) distributions is defined as

\[
p(x|\Theta) = \sum_{i=1}^{m} \pi_i p_i(x|\theta_i),
\]

where \( \Theta = (\pi_1, \ldots, \pi_m, \theta_1, \ldots, \theta_m) \) is the parameter vector and \( p_i(\cdot) \) are the probability densities of the mixture’s components. The weights \( \pi_i \) obey to the following constraint

\[
\sum_{i=1}^{m} \pi_i = 1,
\]

and can be interpreted as the prior probabilities that a given value \( x \) is generated by the component \( i \), i.e. the distribution \( p_i \).

In the case of Gaussian mixtures, the probability density functions \( p_i \) are Gaussian distributions, i.e. univariate or multivariate normal distributions. In this case the parameter is \( \theta_i = (\mu_i, \Sigma_i) \), as normal distributions are defined by the mean and the covariance matrix (or variance). The main challenge when learning mixtures of distributions from the data is the determination of the parameter vector \( \Theta \).

5.3 EM algorithm

The expectation-maximization (EM) algorithm [20] estimates the parameters of the mixture using maximum likelihood. Given a sequence of observations \( X = \{x_1, \cdots, x_n\} \), the likelihood of the probability density function defined in (5.1) is

\[
L(\Theta|X) = \prod_{k=1}^{n} \sum_{i=1}^{m} \pi_i p_i(x_k|\theta_i).
\]

The maximum-likelihood estimate of the parameter is

\[
\Theta^* = \arg \max_{\Theta} L(\Theta|X).
\]

Often the logarithm of the likelihood, i.e. the log-likelihood, is maximized because it is analytically easier. The maximization of the log-likelihood depends on the particular probability density functions that define the mixture. Usually the optimal parameters are found by numerical algorithms as there is no closed form to maximize the likelihood.

The EM algorithm provides a procedure to maximize the likelihood and consequently to find the vector of parameters. It guarantees to find a local optimum. The algorithm
assumes to know from which distribution each observed realization of \( x_k \) is drawn. This information is modeled by considering that each observation consists of two components. The first component is \( x_k \) and the second component is the category \( y_k \), which is a vector of length \( m \) with 1 in the position corresponding to the appropriate category, i.e. to the probability density function. If we indicate with \( z_k \) a complete observation then the complete data is

\[
Z = \{ z_k, \ k = 1, \cdots, n \} = \{ (x_k, y_k), \ k = 1, \cdots, n \}.
\]

Under this assumption the likelihood of the complete data can be calculated

\[
G(\Theta | x_1, \cdots x_n, y_1, \cdots y_n) = \prod_{k=1}^{n} \prod_{i=1}^{m} \pi_{k,i} p_i(x_k | \theta_i)^{y_{k,i}}, \tag{5.2}
\]

where \( y_{k,i} \) is the component \( i \) of the observation vector \( y_k \) which refers to observation \( k \) and is either equal to 1 or to 0. The log-likelihood of the above expression is

\[
\log(G(\Theta | Z)) = \sum_{k=1}^{n} y_k^T v(\pi) + \sum_{k=1}^{n} y_k^T u_k(\Theta), \tag{5.3}
\]

where \( v(\pi) \) and \( u_k(\Theta) \) are vectors of length \( m \) in which the \( i \)-th component is \( \log \pi_i \) and \( \log p_i(x_k | \theta_i) \), respectively. The EM algorithm proceeds as follows. In the expectation step, the expected log-likelihood of the complete data \( Z \) is computed conditioned on the observed (incomplete) data \( X \) and on the previous estimation of the parameter vector \( \Theta^j \). As the expectation is calculated with respect to the incomplete data, the expected log-likelihood \( Q \) depends only on the variable \( \Theta \). In the maximization step, the parameter vector that maximizes the expected conditional log-likelihood is found; this vector is now the new estimate of the parameters. As the process iterates, a sequence of parameter vectors \( \{ \Theta^j \} \) is produced. The algorithm stops when a given condition is satisfied such as a maximum number of iterations reached or an increment in log-likelihood below a given threshold value.

It can be proven [20] that the EM algorithm increases the likelihood of the observed (incomplete) data \( X \) monotonically, i.e. \( \mathcal{L}(\Theta^{j+1} | X) \geq \mathcal{L}(\Theta^j | X) \). However, the algorithms is not guaranteed to converge to a global maximum. In practice, the algorithm can be started from different initial parameter vectors and the best final solution can be selected as an approximation of the global maximum.

From equation (5.3) the expected log-likelihood can be computed as follows:

\[
Q(\Theta, \Theta^j) = \mathbb{E}[\log(G(\Theta | Z)) | X, \Theta^j] = \sum_{k=1}^{n} w_k^T v(\pi) + \sum_{k=1}^{n} w_k^T u_k(\Theta), \tag{5.4}
\]

where

\[
w_k = \mathbb{E}[y_k | x_k, \Theta^j].
\]
Algorithm 5 Generic EM algorithm

Require: input data $\mathcal{X}$, initialize $\Theta^0$ arbitrarily, $j = 0$, define stop-condition

Ensure: $\mathcal{L}(\Theta^{j+1}|\mathcal{X}) \geq \mathcal{L}(\Theta^j|\mathcal{X})$

1: repeat
2: (E step) $Q(\Theta, \Theta^j) = \mathbb{E}[\log(G(\Theta|Z))|\mathcal{X}, \Theta^j]$
3: (M step) $\Theta^{j+1} = \arg\max_\Theta Q(\Theta, \Theta^j)$
4: $j = j + 1$
5: until stop-condition

As the element in column $i$, indicated as $y_{k,i}$, of the vector $y_k$ is equal to 1 if the example $k$ is associated with the category $i$ and 0 otherwise, the expected value is equal to the probability that $y_{k,i}$ is equal to one, i.e.

$$w_{k,i} = \mathbb{E}[y_{k,i}|x_k, \Theta^j] = 0 \cdot p(y_{k,i} = 0|x_k, \Theta^j) + 1 \cdot p(y_{k,i} = 1|x_k, \Theta^j).$$

The quantity $w_{k,i}$ is therefore the probability of category $i$ membership for the observation $k$ given the parameters $\Theta^j$ and the incomplete observation $x_k$. Applying Bayes’s rule we obtain

$$w_{k,i} = p(y_{k,i} = 1|x_k, \Theta^j) = \frac{p(x_k|y_{k,i} = 1, \Theta^j)p(y_{k,i} = 1|\Theta^j)}{p(x_k|\Theta^j)}.$$

Using equation (5.1) we obtain

$$w_{k,i} = \frac{p_i(x_k|\theta_i^j)\pi_i}{p(x_k|\Theta^j)}.$$ (5.5)

As the parameters $\pi_i$ are decoupled from the parameters $\theta_i$ in equation (5.4), we can derive the update rule according to the EM algorithm in order to estimate the prior category probabilities $\pi$, without referring to a particular density function $p_i(x|\theta_i)$. In fact, a necessary condition for a local optimum is that the partial derivatives of the Lagrangian are zero [55].

The Lagrangian $Q_L$ associated with the objective function $Q$ and the constraint $\sum_{i=1}^{m} \pi_i = 1$ is

$$Q_L(\Theta, \Theta^j) = \sum_{k=1}^{n} w_k^T v(\pi) + \sum_{k=1}^{n} w_k^T u_k(\Theta) + \lambda(\sum_{i=1}^{m} \pi_i - 1).$$

By setting the partial derivatives to zero, we obtain

$$\frac{\partial Q_L(\Theta, \Theta^j)}{\partial \pi_i} = \sum_{k=1}^{n} \frac{w_{k,i}}{\pi_i} + \lambda = 0.$$

It can easily be verified that the second partial derivative is always positive, therefore the stationary point is a local maximum. The above equation implies

$$\pi_i = -\sum_{k=1}^{n} \frac{w_{k,i}}{\lambda}.$$
In order to find the value of $\lambda$, we sum the above equation over all possible $i$ and obtain

$$\sum_{i=1}^{m} \pi_i = -\sum_{i=1}^{m} \sum_{k=1}^{n} \frac{w_{k,i}}{\lambda}.$$ 

This implies

$$\lambda \sum_{i=1}^{m} \pi_i = -\sum_{k=1}^{n} \sum_{i=1}^{m} w_{k,i} \rightarrow \lambda = -n.$$ 

By substituting $\lambda$ we finally find the update rule for the parameters $\pi_i$:

$$\pi_i = \frac{1}{n} \sum_{k=1}^{n} w_{k,i}.$$ 

Note that the weights $w_{k,i}$ are defined in equation (5.5) and depend on the parameters estimated in the preceding step and on the particular probability density functions that define the mixture (5.1).

In the case of Gaussian mixtures, we can apply the EM algorithm to estimate the remaining parameters $\theta_i = (\mu_i, \Sigma_i)$ for $i \in [1, m]$. The component $i$ of a $d$-dimensional Gaussian mixture has the following probability density function:

$$p_i(x|\theta_i) = p_i(x|\mu_i, \Sigma_i) = \frac{1}{(2\pi)^{\frac{d}{2}} |\Sigma_i|^{\frac{1}{2}}} e^{-\frac{1}{2}(x-\mu_i)^T \Sigma_i^{-1} (x-\mu_i)},$$

where we indicate the determinant of matrix $\Sigma_i$ with $|\Sigma_i|$.

To estimate the mean $\mu_i$ and the covariance matrix $\Sigma_i$, we maximize the expected log-likelihood (5.4). As the partial derivatives of $\pi_i$ and $\theta_i$ are decoupled, we need to maximize only the second addend:

$$\sum_{k=1}^{n} w_k^T u_k(\Theta) = \sum_{k=1}^{n} \sum_{i=1}^{m} w_{k,i} u_{k,i}(\Theta) = \sum_{k=1}^{n} \sum_{i=1}^{m} w_{k,i} \log(p_i(x_k|\theta_i)).$$

The above quantity is equal to

$$\sum_{k=1}^{n} \sum_{i=1}^{m} w_{k,i} \left(-\frac{d}{2} \log(2\pi) - \frac{1}{2} \log(|\Sigma_i|) - \frac{1}{2} (x_k - \mu_i)^T \Sigma_i^{-1} (x_k - \mu_i) \right).$$

Therefore by setting to zero the partial derivative of the expected log-likelihood with respect to $\mu_i$, we obtain

$$\frac{\partial Q(\Theta, \Theta^j)}{\partial \mu_i} = -\sum_{k=1}^{n} w_{k,i} \Sigma_i^{-1} (x_k - \mu_i) = 0.$$ 

\footnote{Note that for a matrix $A$ and a vector $x$ the derivative $\frac{\partial x^T A x}{\partial x} = (A + A^T)x$, it follows that if $A$ is symmetric, as the covariance matrix $\Sigma$ and its inverse $\Sigma^{-1}$, the derivative is equal to $2Ax$.}
5.4. Estimating conditional distributions

It can be verified that the second partial derivative is a positive semi-definite matrix, therefore the stationary point is a local maximum. The solution of the above equation is

\[ \mu_i = \frac{\sum_{k=1}^{n} w_{k,i} x_k}{\sum_{k=1}^{n} w_{k,i}}. \]

The estimator of the covariance matrix is found by setting the partial derivative of \( Q \) with respect to \( \Sigma_i \) to zero, i.e.

\[ \frac{\partial Q(\Theta, \Theta^j)}{\partial \Sigma_i} = 0. \]

As the calculation in this case is quite complex I do not report it here, an easy to read version is described in [9]. By solving the above equation, we obtain the following estimate of the covariance matrix

\[ \Sigma_i = \frac{\sum_{k=1}^{n} w_{k,i} (x_k - \mu_i)(x_k - \mu_i)^T}{\sum_{k=1}^{n} w_{k,i}}. \]

Now we can summarize and describe the EM algorithm in the case of Gaussian mixtures.

**Algorithm 6** EM algorithm for Gaussian mixtures

**Require:** input data \( \mathcal{X} \), initialize \( \mu_i \) and \( \Sigma_i \) arbitrarily \( \forall i \in [1, m] \), define *stop-condition*

**Ensure:** \( \mathcal{L}(\Theta^{j+1}|\mathcal{X}) \geq \mathcal{L}(\Theta^j|\mathcal{X}) \)

1: repeat
2: for \( i = 1 \) to \( m \) do
3: for \( k = 1 \) to \( n \) do
4: \[ w_{k,i} = \frac{\pi_i p_i(x_k|\mu_i, \Sigma_i)}{\sum_{j=1}^{m} \pi_j p_j(x_k|\mu_j, \Sigma_j)} \]
5: end for
6: \[ \pi_i = \frac{1}{n} \sum_{k=1}^{n} w_{k,i} \]
7: \[ \mu_i = \frac{\sum_{k=1}^{n} w_{k,i} x_k}{\sum_{k=1}^{n} w_{k,i}} \]
8: \[ \Sigma_i = \frac{\sum_{k=1}^{n} w_{k,i} (x_k - \mu_i)(x_k - \mu_i)^T}{\sum_{k=1}^{n} w_{k,i}} \]
9: end for
10: until *stop-condition*

5.4 Estimating conditional distributions

Gaussian mixtures allow us to model conditional distribution which can be used for prediction. For instance, the expected value or the mode of the distribution can be taken as the prediction given the input features. Assume that the data is \( d \)-dimensional, the observation \( k \) then is:

\[ z_k = \{ z_{k,1}, \ldots, z_{k,d} \}. \]
A Gaussian mixture can model the joint probability density function in the \(d\)-dimensional space. By looking at the data as a learning set, where the component \(z_{k,d}\) is the output and the remaining elements are the input features, i.e.

\[
z_{k,d} = f(z_{k,1}, \cdots, z_{k,d-1}),
\]

the relationship between input and outputs can be modeled using a mixture of Gaussian distributions. As the relationship \(f\) is in general not deterministic, the most general way to model the dependence between output and input is by finding the conditional distribution of \(z_{k,d}\) given \(z_{k,1}, \cdots, z_{k,d-1}\). For a particular set of input values, the expected value of the conditional distribution can be taken as the output prediction. Using the expected value of the conditional distribution, the mean squared prediction error is minimized [37].

To generalize, the observed data can be decomposed into two parts: the vector of input features \(x_k\) and the vector of outputs \(y_k\), i.e.

\[
z_k = \{z_{k,1}, \cdots, z_{k,d-i}, z_{k,d-i+1}, \cdots, z_{k,d}\}.
\]

If \(i = 1\) then the output is a scalar. Once the conditional probability density function \(p(y|x)\) is known, the minimum mean squared error (MSE) estimator is given by

\[
\hat{y} = \mathbb{E}[y|x] = \int_{\mathcal{Y}} y p(y|x) dy,
\]

where \(\mathcal{Y}\) is the domain of the vector \(y\). The maximum a posteriori (MAP) estimator is given by

\[
\hat{y} = \arg \max_y p(y|x).
\]

The conditional distribution therefore plays a key role in the prediction of outputs. Moreover the conditional distribution provides information about the uncertainty of the prediction as the stochastic relationship between input and output is explicitly modeled. Next I discuss how to build a conditional distribution from a learning set using a mixture of Gaussian distributions.

Assume the learning set is \(\mathcal{Z} = \{\mathcal{X}, \mathcal{Y}\}\), where \(x \in \mathcal{X}\) are the input features and \(y \in \mathcal{Y}\) are the outputs. The joint probability function \(p(z) = p(x, y)\) can be estimated using the mixture model (5.1):

\[
p(z) = p(x, y|\Theta) = \sum_{i=1}^{m} \pi_i p_i(x, y|\theta_i).
\]

In the following I omit the parameters \(\Theta\) and \(\theta_i\) from the notation for the sake of simplicity. Using Bayes’s rule, we obtain

\[
p(y|x) = \frac{p(x, y)}{p(x)} = \frac{1}{p(x)} \sum_{i=1}^{m} \pi_i p_i(x, y) = \frac{1}{p(x)} \sum_{i=1}^{m} \pi_i p_i(y|x) p_i(x).
\]
In the case of Gaussian mixtures, if the parameters of the joint distribution \( p(x, y) \) are known, then the marginal densities \( p(x) \), \( p_i(x) \), and \( p_i(y|x) \) can easily be derived in closed form as follows. The marginal density \( p_i(x) \) of component \( i \) is given by extracting the mean \( \mu_{x,i} \) and the covariance matrix \( \Sigma_{xx,i} \) from the parameters of the component [19]. Therefore if \( p_i(x, y) \sim \mathcal{N}(\mu_i, \Sigma_i) \), where
\[
\mu_i = [\mu_{x,i}, \mu_{y,i}]^T,
\]
and
\[
\Sigma_i = \begin{pmatrix} \Sigma_{xx,i} & \Sigma_{xy,i} \\ \Sigma_{yx,i} & \Sigma_{yy,i} \end{pmatrix},
\]
then \( p_i(x) \sim \mathcal{N}(\mu_{x,i}, \Sigma_{xx,i}) \). The marginal distribution \( p(x) \) is then given by
\[
p(x) = \sum_{i=1}^{m} \pi_i p_i(x).
\]
Finally, the conditional density of each component \( p_i(y|x) \) is a Gaussian with the following parameters [19]:
\[
p_i(y|x) \sim \mathcal{N}(\mu_{y,i} + \Sigma_{yx,i} \Sigma_{xx,i}^{-1}(x - \mu_{x,i}), \Sigma_{yy,i} - \Sigma_{yx,i} \Sigma_{xx,i}^{-1} \Sigma_{yx,i}).
\] (5.6)
The conditional probability density function \( p(y|x) \) is therefore a Gaussian mixture itself. In fact, for a given value of \( x \) the weights are
\[
\pi_{x,i} = \frac{\pi_i p_i(x)}{p(x)},
\]
and the mixture is defined by
\[
p(y|x) = \sum_{i=1}^{m} \pi_{x,i} p_i(y|x).
\] (5.7)
Once the joint density \( p(x, y) \) has been estimated using the EM algorithm, the conditional density \( p(y|x) \) can therefore easily be computed.

5.5 Customer value distributions

Using the methods described in the preceding section, the distribution of customer value can be estimated given some known features characterizing the customer. Of course, I assume that a training set is available from which the stochastic relationship between future value and some input feature such as age, revenue in the last three months, frequency of transactions in the last three months, etc., can be estimated.

Once that the individual value distributions have been predicted, a function mapping distributions to numerical values can be used to define the utility of a customers. In this
way the utility theory framework can be used to establish customer portfolios that maximize
the value while minimizing the risk due to the uncertainty about the future value of the
customers.

The customer lifetime value distribution can be estimated as follows. Assume that the
training set \( \{(x_k, clv_k), \ k = 1, \ldots, n\} \) is available, where the vector \( x_k \) contains different
numerical features characterizing the customer \( k \) and \( clv_k \) represents the future lifetime value
of customer \( k \). The future lifetime value can be observed from historical data by setting the
reference point back into the past, e.g. one year back, and then computing the cumulative
value generated by each customer from the reference point to the future. From the above
training set the joint density \( p(clv, x) \) can be estimated using a mixture of Gaussians and the
EM algorithm. Once the parameters of the mixture are known, the conditional distribution
\( p(clv|x) \) is obtained as a mixture of Gaussians by computing the new priors \( \pi_{x,i} \) and the new
Gaussian components \( p_i(clv|x) \) as described in section 5.4. If the customer value distribution
is known, the future value of a customer described by the input features \( x \) can be predicted
as follows:

\[
\hat{clv} = E[clv|x].
\]

In the case of mixtures of the form (5.7), if we indicate the mean of the \( i \)-th component with
\( \mu_{x,i} \), the expected value is

\[
\hat{clv} = E[\sum_{i=1}^{m} \pi_{x,i} p_i(clv|x)] = \sum_{i=1}^{m} \pi_{x,i} \mu_{x,i},
\]

and using formula (5.6), we derive

\[
\mu_{x,i} = \mu_{clv,i} + \Sigma_{clvx,i} \Sigma_{xx,i}^{-1} (x - \mu_{x,i}).
\]

The parameters \( \mu_{clv,i}, \Sigma_{clvx,i}, \Sigma_{xx,i}, \) and \( \mu_{x,i} \) are estimated using the EM algorithm. The
algorithm 7 summarizes the entire procedure.

5.6 Customer utility and portfolio selection

The expected value of a distribution provides an estimate of the average outcome if a large
number of samples are drawn from the distribution. If the number of samples is small, e.g.
two or three, then the observed values can differ significantly from the expected value, espe-
cially if the variance of the distribution is large. Therefore if we associate to each customer
the expected value of his lifetime value distribution, we lose information concerning the
uncertainty (i.e. the risk) involved in the prediction.

Utility theory (see appendix A) provides an analytic framework for the selection of al-
ternatives while taking into account the uncertainties of the possible outcomes. Customers
Algorithm 7 Prediction of customer lifetime value distribution

Require: input data \( \{z_k = (x_k, clv_k), k = 1, \ldots, n\} \), set number of components of the mixture to \( m \), initialize \( \mu_i \) and \( \Sigma_i \) arbitrarily \( \forall i \in [1, m] \), define stop-condition

Ensure: \( p(clv|x) \) and \( \hat{clv} \)

1: repeat
2: \hspace{1em} for \( i = 1 \) to \( m \) do
3: \hspace{2em} for \( k = 1 \) to \( n \) do
4: \hspace{3em} \( w_{k,i} = \frac{\pi_i p_i(z_k | \mu_i, \Sigma_i)}{\sum_{i=1}^m \pi_i p_i(z_k | \mu_i, \Sigma_i)} \)
5: \hspace{2em} end for
6: \hspace{2em} \( \pi_i = \frac{1}{n} \sum_{k=1}^n w_{k,i} \)
7: \hspace{2em} \( \mu_i = \frac{\sum_{k=1}^n w_{k,i} x_k}{\sum_{k=1}^n w_{k,i}} \)
8: \hspace{2em} \( \Sigma_i = \frac{\sum_{k=1}^n w_{k,i} (x_k - \mu_i)(x_k - \mu_i)^T}{\sum_{k=1}^n w_{k,i}} \)
9: \hspace{2em} end for
10: until stop-condition
11: for \( i = 1 \) to \( m \) do
12: \hspace{2em} decompose \( \mu_i = [\mu_{x,i}, \mu_{clv,i}]^T \)
13: \hspace{2em} decompose \( \Sigma_i = \begin{pmatrix} \Sigma_{xx,i} & \Sigma_{xclv,i} \\ \Sigma_{clvx,i} & \Sigma_{clvclv,i} \end{pmatrix} \)
14: \hspace{2em} \( p_i(x) \sim N(\mu_{x,i}, \Sigma_{xx,i}) \)
15: \hspace{2em} \( p_i(clv|x) \sim N(\mu_{clv,i} + \Sigma_{clvx,i}^{-1}(x - \mu_{x,i}), \Sigma_{clvclv,i} - \Sigma_{clvx,i}^{-1}\Sigma_{xx,i}\Sigma_{clvx,i}) \)
16: \hspace{2em} end for
17: for \( i = 1 \) to \( m \) do
18: \hspace{2em} \( \pi_{x,i} = \frac{\pi_i p_i(x)}{\sum_{i=1}^m \pi_i p_i(x)} \)
19: \hspace{2em} end for
20: \hspace{2em} \( p(clv|x) = \sum_{i=1}^m \pi_{x,i} p_i(clv|x) \)
21: \hspace{2em} \( \hat{clv} = \mathbb{E}[clv|x] = \sum_{i=1}^m \pi_{x,i} (\mu_{clv,i} + \Sigma_{clvx,i}^{-1}(x - \mu_{x,i})) \)

5.6. Customer utility and portfolio selection

can be ranked based on their expected utility rather than on their expected value. In this way, marketing managers can express their preferences for certain distribution profiles, e.g. a risk-averse decision maker can select customers for whom the uncertainty about the future expected value is not too large. In the case of aggressive marketing strategies, e.g. expansion into a new market, marketing managers can exhibit risk-seeking behavior, leading to a preference for customers with a very risky future value distribution, in which the uncertainty about the future outcome is large.

Given a utility function \( u(\cdot) \), the expected utility of a mixture of Gaussians is

\[
\mathbb{E}[u(clv)|x] = \sum_{i=1}^m \mathbb{E}[u(clv)|x, i]p(i|x) = \sum_{i=1}^m u_{x,i} \pi_{x,i}, \tag{5.8}
\]

where \( u_{x,i} \) is the expected utility of the \( i \)-th component of the mixture (5.7). If \( y \) is normally
distributed, i.e. $y \sim \mathcal{N}(\mu_y, \sigma_y^2)$, and constant risk aversion is assumed, i.e. the utility function is exponential of the form $u(y) = -e^{-cy}$ with $c > 0$ [43], then the expected utility is

$$\mathbb{E}[u(y)] = -e^{-cy} + \frac{c^2}{2} \sigma_y^2.$$  

It follows that in the case of the Gaussian mixture

$$u_{x,i} = -e^{-c(\mu_{clv,i} + \Sigma_{clv,i}^{-1}(x - \mu_{x,i})) + \frac{c^2}{2} (\Sigma_{clv,i}^{-1} - \Sigma_{clv,i}^{-1} \Sigma_{x,i} \Sigma_{clv,i}^{-1})},$$

for constant risk proneness, i.e. $u(y) = e^{-cy}$ with $c < 0$, we obtain

$$u_{x,i} = e^{-c(\mu_{clv,i} + \Sigma_{clv,i}^{-1}(x - \mu_{x,i})) + \frac{c^2}{2} (\Sigma_{clv,i}^{-1} - \Sigma_{clv,i}^{-1} \Sigma_{x,i} \Sigma_{clv,i}^{-1})}.$$  

Next I consider the case of constant risk aversion, which implies a utility function of the form $u(y) = a - be^{-cy}$. The following property holds in this case.

**Property 5.6.1 (Certain equivalent).** The certain equivalent (CE) under the assumption of an exponential utility function of the form $u(y) = a - be^{-cy}$ is given by

$$CE = -\frac{1}{c} \log(\mathbb{E}[e^{-cy}]).$$

**Proof.** The definition of certain equivalent implies that $u(CE) = \mathbb{E}[u(y)]$, therefore

$$u(CE) = a - be^{-cCE} = \mathbb{E}[a - be^{-cy}] = a - b\mathbb{E}[e^{-cy}].$$

By solving the above equation we obtain

$$CE = -\frac{1}{c} \log(\mathbb{E}[e^{-cy}]).$$

The next property refers to a portfolio of independent alternatives.

**Theorem 5.6.1 (Portfolio of independent alternatives).** Given an exponential utility function of the form $u(y) = a - be^{-cy}$, the certainty equivalent $CE$ of a portfolio $p$ of $n$ independent alternatives $y_1, \cdots, y_n$ defined as

$$p = \sum_{i=1}^{n} y_i,$$

is equal to the sum of the certainty equivalents of the different alternatives $CE_i$, i.e.

$$CE = \sum_{i=1}^{n} CE_i.$$
Proof. Using property 5.6.1 the certainty equivalent of the portfolio is

\[ CE = -\frac{1}{c} \log(\mathbb{E}[e^{-cp}]) = -\frac{1}{c} \log(\mathbb{E}[e^{-c\sum_{i=1}^{n} y_i}]) = -\frac{1}{c} \log(\mathbb{E}\prod_{i=1}^{n} e^{-cy_i}). \]

If the alternatives \( y_i \) are independent then

\[ -\frac{1}{c} \log(\mathbb{E}\prod_{i=1}^{n} e^{-cy_i}) = -\frac{1}{c} \log(\prod_{i=1}^{n} \mathbb{E}[e^{-cy_i}]) = \sum_{i=1}^{n} -\frac{1}{c} \log(\mathbb{E}[e^{-cy_i}]). \]

As the certainty equivalent of alternative \( y_i \) is

\[ CE_i = -\frac{1}{c} \log(\mathbb{E}[e^{-cy_i}]), \]

we obtain \( CE = \sum_{i=1}^{n} CE_i. \)

Assuming customers are independent, I can use the above property to build portfolios of customers under a marketing budget constraint. As \( u(CE) = \mathbb{E}[u(y)] \), the expected utility is maximized if the utility of the certainty equivalent \( CE \) is maximized because the utility function is monotonically increasing, i.e. \( u'(\cdot) > 0 \). This is equivalent to maximizing the certainty equivalent. Therefore, according to the Expected Utility Theorem, the customer portfolio that maximizes the certainty equivalent should be selected.

The optimization problem is therefore to find the customers that maximize the certainty equivalent, or equivalently the expected utility, of the resulting portfolio while satisfying the budget constraints. As the number of customers is discrete, this problem is a linear integer optimization problem. In practice, in a direct marketing context, the number of customers is large and the cost to target an individual customer is not too high compared with the total available marketing budget. I can therefore solve the equivalent continuous optimization problem by assuming that a fractional number of customers can be targeted and then rounding this number to an integer. The equivalent continuous optimization problem can be simply solved by ranking all the customers according to their certainty equivalent, or equivalently according to their expected utility, and then allocating resources until the marketing budget is exhausted. Of course only those customers with a positive return on investment should be added. To apply this procedure the utility has to be calculated on the return on marketing investment. The entire procedure is summarized in algorithm 8.

The “rank and cut” method is quite well known in marketing practice [63]; customers are scored according to their expected value or expected probability to respond to a marketing campaign. The developed approach extends the standard practice because the uncertainty about the future prediction is explicitly taken into account by using utility theory. Moreover, the approach generalizes the standard practice because in the case of risk neutrality of the decision maker, the expected utility is equal to the expected value.
Algorithm 8 Optimal customer portfolio maximizing the value-risk tradeoff

**Require:** utility function \( u(y) = a - be^{-cy} \), expected utility of the customer computed using algorithm 7 and equation (5.8), customer set \( C \), marketing budget \( b \), cost to target customer \( i: cost_i \)

**Ensure:** customer portfolio \( P \)

1: rank all customer in \( C \) according to their expected utility in descending order, let \( customer_i \) be the customer with rank \( i \)
2: \( P = \{\}, i=0 \)
3: repeat
4: \( P = P \cup \{customer_i\} \)
5: \( b = b - cost_i \)
6: \( i=i+1 \)
7: until \( b \leq 0 \)

5.7 Case study

In this section I describe a numeric example illustrating how the customer lifetime value distribution can be predicted using Gaussian mixtures and how a portfolio of customers is built by ranking customers according to their expected utility. I use the customer transaction data of a leading European airline to estimate the value distribution at an individual level for each customer in the database.

5.7.1 Input data

The transaction histories of approximately 450000 customers is stored in the airline’s database. The transactions refer to a time period of four years, from 2000 to 2004. To build the training and test sets I consider all the customers in the database who made at least one transaction before the 15th of September 2001 (past); these are the active customers. Then I compute their features in the next 12 months. The 15th of September 2002 is considered the reference data (present). To track eventual changes in the buying behavior I compute features referring to the last three, last six, and last twelve months with respect to the reference date. I compute the following features:

- Value generated in the last twelve months (value12),
- number of transactions effectuated in the last twelve months (freq12),
- value generated in the last six months (value6),
- number of transactions effectuated in the last six months (freq6),
• value generated in the last three months (value3),
• number of transactions effectuated in the last three months (freq3), and
• age of the customer (referring to the present).

Then I compute the future lifetime value, defined as the cumulative revenue generated by each customer from the present until the 15th of September 2003 (future). Figure 5.1 shows the time window considered.

![Figure 5.1: The time window for the computation of the training set.](image)

I obtain a set containing more than 20000 customers. As this is a large number of customers and performing all the calculations required for the estimation of the value distribution is quite intensive, I randomly extract 9122 customers from this set. Table 5.1 summarizes the descriptive statistics of the customer set obtained in this way. Note that the maximum future value is above three millions; this suggests that there are outliers.

<table>
<thead>
<tr>
<th>Feature</th>
<th>mean ($\mu$)</th>
<th>standard deviation ($\sigma$)</th>
<th>maximum</th>
<th>minimum</th>
</tr>
</thead>
<tbody>
<tr>
<td>value</td>
<td>10434.4</td>
<td>68899.6</td>
<td>3654673.5</td>
<td>0</td>
</tr>
<tr>
<td>value3</td>
<td>2190.1</td>
<td>29100.8</td>
<td>1663217</td>
<td>0</td>
</tr>
<tr>
<td>value6</td>
<td>4347.9</td>
<td>37371.7</td>
<td>1663549</td>
<td>0</td>
</tr>
<tr>
<td>value12</td>
<td>6995.6</td>
<td>39193.5</td>
<td>1663683.5</td>
<td>0</td>
</tr>
<tr>
<td>freq3</td>
<td>4.4</td>
<td>5.5</td>
<td>47</td>
<td>0</td>
</tr>
<tr>
<td>freq6</td>
<td>11</td>
<td>12.2</td>
<td>114</td>
<td>0</td>
</tr>
<tr>
<td>freq12</td>
<td>23.8</td>
<td>25.4</td>
<td>225</td>
<td>0</td>
</tr>
<tr>
<td>age</td>
<td>45.6</td>
<td>10.5</td>
<td>92</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 5.1: Descriptive statistics of the features.

Outliers can be due to errors in the records of the database or due to corporate customers, i.e. companies, that register all the travel expenses of their employees into one account. As the goal is to forecast the distribution of the individual customers, I remove the outliers as
follows. I define an outlier as a customer whose value in the next twelve months will exceed the average future value plus two times the standard deviation, i.e.

\[ \text{value} > \mu + 2\sigma. \]

In this way I remove 65 outliers. If I use a higher threshold and define as outliers customers who exceed the mean plus three times the standard deviation I obtain almost the same data set. Now that the data set is built, I extract from it randomly 60% of the customers to build the training set; the remainder of the data constitutes the test set used to validate the model. The training set consists of 5362 customers, whereas the test set consists of 3695 customers.

5.7.2 Model training and testing

I estimate the mixture of Gaussians modeling the joint probability density distribution between the input features and the observed future lifetime value according to [4]. To estimate the right number of components of the mixture, an initial model is built with only one Gaussian component. Then different heuristics are applied in order to increase the number of nodes of the mixture. One node is split into two nodes if the kurtosis is higher than a given threshold. A high kurtosis indicates in fact that the node does not follow a Gaussian distribution, therefore splitting the node could increase the likelihood on the data. Other operators include the pruning and merging of nodes. Nodes with a very low weight are pruned from the mixture model, and very similar nodes are merged into one node. All these operators are described in [4]. A maximum number of steps is specified before running the EM algorithm. If the number of iterations reaches the specified maximum number of steps, or if the likelihood does not increase significantly, the iterative algorithm stops.

Other methods could be used to find the best number of mixtures. A general approach could be to build several models and then select the one that best fits the data according to the likelihood. Cross-validation [37] can be used to improve the quality of the estimated performance measure (i.e. likelihood) of each model. However, the disadvantage of such a procedure is the huge number of calculations required. Heuristics are therefore preferred in many practical situations.

Figure 5.2 shows the increase of the log-likelihood and of the number of components during the training phase. I start with one component, and the maximum number of iterations is 150. However, the algorithm already converges after 97 iterations. The final mixture model has 32 components. As the EM algorithm is guaranteed to converge to a local maximum, I repeated the procedure several times with several initial values of the parameters and obtained very similar final mixtures (in terms of likelihood and number of components). One can conclude that the learned model is quite robust.
To test the predictive accuracy of the learned mixture, I use an independent test set, that has not been used in the learning phase. As I cannot observe the distribution of the future value, I compare the predicted future value \( \hat{clv} \) with the observed future lifetime value. There are several indicators to measure the performance of a predictive model. The root mean squared error (MSE) is defined as

\[
\sqrt{MSE} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (clv_i - \hat{clv}_i)^2},
\]

where \( clv_i \) is the observed and \( \hat{clv}_i \) the predicted value of customer \( i \); \( n \) is the number of samples in the test set. The absolute mean error (AME) is defined as

\[
AME = \frac{1}{n} \sum_{i=1}^{n} |clv_i - \hat{clv}_i|.
\]

The root mean squared error penalizes large deviations between the predicted and the observed value more than the absolute mean error does because of the square operation. For this reason, I consider the absolute mean error as a better indicator to measure the performance of the model on the test data. Other derived indicators are the relative root mean squared error (RRMSE) and the relative absolute mean error (RAME), defined respectively
as the ratio between the error indicator measured using the model and the error indicator measured using a trivial predictor (i.e. the average value calculated on the training set). The trivial predictor therefore always predicts the (same) value for each instance on the test set, without considering the input features. Analytically the definitions are the following.

$$RRMSE = \sqrt{\frac{MSE_{model}}{MSE_{trivial}}}$$

and

$$RAME = \frac{AME_{model}}{AME_{trivial}}.$$  

Both these relative errors should be less than one to justify the use of a complex predictive model. In fact, a relative error that is larger than one indicates that the trivial predictor, i.e. the expected value, performs better than the model. Another indicator of the quality of the predictive model is the correlation coefficient between the predictions and the observed future value.

To assess the quality of the model, I compare the above-defined error indicators with those of state-of-the-art predictive algorithms. I provide the same training and testing sets to each algorithm\(^2\). Table 5.2 summarizes the results.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>RAME</th>
<th>RRMSE</th>
<th>root MSE</th>
<th>correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian mixture (32 components)</td>
<td>0.7046</td>
<td>0.9683</td>
<td>17286</td>
<td>0.3518</td>
</tr>
<tr>
<td>M5 model tree</td>
<td>0.7452</td>
<td>1.0394</td>
<td>18555</td>
<td>0.2812</td>
</tr>
<tr>
<td>Linear regression</td>
<td>0.7664</td>
<td>0.9935</td>
<td>17735.72</td>
<td>0.2551</td>
</tr>
<tr>
<td>1-nearest-neighbor</td>
<td>0.7770</td>
<td>1.1924</td>
<td>21287.61</td>
<td>0.3238</td>
</tr>
<tr>
<td>Neural network, 1 hidden layer, 4 sigmoids, 500 epochs</td>
<td>0.8791</td>
<td>1.1705</td>
<td>20896.87</td>
<td>0.1289</td>
</tr>
</tbody>
</table>

Table 5.2: Performance benchmark for the prediction of customer lifetime value.

The models are ranked according to the relative absolute mean error. According to this ranking, the Gaussian mixture models performs best, i.e. 30% better than the trivial (average on training set) predictor. However, the differences to the other predictive algorithms are not so large. One can conclude that predicting lifetime value is a difficult task, in fact all the models exhibit quite a poor performance, especially if measured using the relative root squared mean error. Therefore rather than claiming that the Gaussian mixture performs better than the state-of-the-art algorithms, I claim that its performance is in line with that of the other methods.

\(^2\)I will not describe each algorithm in detail because the purpose is to provide a benchmark to compare the mixture of Gaussians model with other predictive models. A detailed description of the algorithms used can be found in [37], [24], [88].
5.7. Case study

5.7.3 Customer value distributions

Once the mixture density has been obtained, I apply all the steps described in algorithm 7 in order to predict the customers’ future value distributions and consequently the customers’ future value by computing the expectation of the value distribution. Figure 5.3 shows the value distributions of four different customers, identified through their position in the test file.

![Figure 5.3: Example of lifetime value distributions of four customers.](image)

Note that the customers have different distributions, the predicted value (expected value of the distribution) is indicated with the symbol $\times$ whereas the observed value is indicated with the symbol $\circ$.

By comparing the distributions of customer 3006 and of customer 2 one can note that the first distribution has a higher expected value while the second distribution has a fatter right tail. Therefore the two distributions have different value-risk profiles, and risk-seeking decision maker would prefer customer 2, because of a higher probability of observing a high outcome, whereas a risk-averse or risk-neutral decision maker would prefer customer 3006. It is also interesting that the observed value is higher for customer 2; a risk-seeking choice had therefore led to a higher return.

By comparing the distributions of customers 123 and 26 one can note that the expected
values are almost the same but the first customer exhibits a higher risk than the second. In fact, the tails of the distribution of the customer 123 are longer and the probability of extreme outcomes is therefore higher. Again, a risk-seeking investor would prefer the distribution 123, whereas a risk-averse decision maker would prefer the distribution 26.

Finally, one can note that even if the training set has only positive future values, the distributions show that there is a probability of observing a future negative value. The reason for this apparently unrealistic outcome is that the Gaussian mixture extrapolates from the available data in order to fit a distribution to a wider range of values. Moreover, the components of the mixture are normal densities, which are symmetric, therefore a range of values that is on the left of some observed value can be included. For the same reason it is possible to estimate an entire distribution for a given customer even if only one transaction of that customer has been observed in the database.

The obtained value distribution allows us to model the probability that a customer generates a value smaller than an arbitrarily value \( \epsilon \geq 0 \) in the next twelve months. This probability can be interpreted as the probability of defection, i.e.

\[
p(\text{defection}) = p(clv \leq \epsilon).
\]

Computing the above probability is easy once the mixture of Gaussians estimating the value distribution is known, in fact:

\[
p(clv \leq \epsilon | x) = \sum_{i=1}^{m} \pi_{x,i} p_{i}(clv \leq \epsilon | x).
\]

The cumulative probability function of the univariate normal distributions

\[ p_{i}(clv \leq \epsilon | x) \]

can easily be computed as it only requires knowledge of the cumulative probability function of the standard normal distribution, which is implemented in most statistical software packages and available in tabular representation in almost any statistics textbook.

To predict whether a customer is a defector, one needs to fix the value of \( \epsilon \) and to find a threshold \( \hat{p} \) so that the following rule can be applied: if a customer has a defection probability higher than \( \hat{p} \) then classify him as a defector, else as a loyal customer. To find the best values of \( \epsilon \) and \( \hat{p} \), one can test different rules and keep the rule that has the best performance, e.g. minimum misclassification error measured using cross-validation [37].

If for instance the following parameters are used:

\[
\epsilon = 50,
\]

and

\[
\hat{p} = 0.85,
\]
one can classify customers into loyal and defectors. To assess the performance of this rule, one can use the *misclassification error* defined as the total number of misclassified customers divided by the total number of customers.

As in the case of the prediction of the lifetime value, I compare the performance of the Gaussian mixture (GM) classifier with that of different state-of-the-art classification algorithms. I provide the same training and testing sets to all algorithms.

In the case of classification, one can decompose the total number of misclassified instances into false positives (FP) and false negatives (FN). The former are customers predicted to be defectors who in reality are not defectors, the latter are customers who are predicted to be loyal (not defectors) but are defectors. In the same way, the correctly classified customers can be divided into true positives (TP), i.e. customers who are predicted to defect and that defect in reality, and true negatives, i.e. customers that are predicted to be loyal and are loyal in reality. The performance of a classification algorithm can be measured by taking into account the costs of the two types of misclassifications. Table 5.3 reports the different types of errors. The Gaussian mixture classifier performs best according to the absolute misclassification rate. However as the two classes (defectors, loyal) are extremely unbalanced, the misclassification error might not be the appropriate performance measure [88].

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>misclassification error</th>
<th>FP</th>
<th>FN</th>
<th>TP</th>
<th>TN</th>
</tr>
</thead>
<tbody>
<tr>
<td>GM classifier</td>
<td>0.1337</td>
<td>169</td>
<td>325</td>
<td>274</td>
<td>2927</td>
</tr>
<tr>
<td>C4.5 tree</td>
<td>0.1372</td>
<td>206</td>
<td>301</td>
<td>298</td>
<td>2890</td>
</tr>
<tr>
<td>Logistic regression</td>
<td>0.1399</td>
<td>107</td>
<td>410</td>
<td>189</td>
<td>2989</td>
</tr>
<tr>
<td>Naive Bayes</td>
<td>0.4974</td>
<td>1822</td>
<td>16</td>
<td>583</td>
<td>1274</td>
</tr>
<tr>
<td>Neural network, 1 hidden layer, 4 sigmoids, 500 epochs</td>
<td>0.1553</td>
<td>326</td>
<td>248</td>
<td>351</td>
<td>2770</td>
</tr>
</tbody>
</table>

Table 5.3: Performance benchmark for the prediction of defectors.

More realistic performance measures should take into account different misclassification costs between false positives and false negatives or the lift curve [17] representing the percentage of defectors versus the percentage of the total number of customers. A lift curve is built by ranking all customers according to the probability of being defectors. On the $x$-axis the cumulative percentage of customers is represented and on the $y$-axis the cumulative percentage of real defectors (positive examples) observed in the testing set is represented. In this way it is possible to see the percentage of the defectors found in a given sub-sample of the customer set, e.g. the first 10% of the customers. Figure 5.4 shows the lift curves for the different algorithms.
The neural network and the logistic regression have almost the same lift curve. At the point corresponding to the first 29% of the customers, the performance of the Naive Bayes classifier improves whereas that of the Gaussian mixture and of the C4.5 decision tree decreases. The best performance, as measured by the lift, is achieved by the logistic regression and the neural network classifiers. One can conclude that the performance of the Gaussian mixture classifier is in line with the performance of the best classification algorithms if one considers the problem of identifying the defectors among a subset that has the size of 29% of the entire population. This subset allows us to identify approximately 76% of all the defectors. In practice, if the budget constraint allows us to target at most 29% of the population with marketing actions aimed at preventing defection, the performance of Gaussian mixture is equivalent to the performance of logistic regression, of neural networks and of C4.5 classification trees. Note that the parameters $\epsilon$ and $\hat{p}$ of the Gaussian mixture classifier have been chosen using common sense; a systematic search using cross-validation could have led to a classification rule with a better performance.
5.7.4 Customer utility and portfolio selection

I calculate the utility of each customer using a risk-averse exponential utility function of the form \( u(x) = -e^{-cx} \). The risk-aversion coefficient \( c \) can be determined using the estimation techniques described in appendix A.

Before applying formula (5.8) I need to normalize all the data. In fact, there are numerical problems when the expected utility of a very large value is calculated. Recall that the expected utility of a Gaussian component with mean \( \mu \) and variance \( \sigma^2 \) is

\[
E[u(\cdot)] = -e^{-c\mu + 0.5c^2\sigma^2}.
\]

If \( \sigma^2 \) is very large, then the expected utility diverges towards minus infinitive, and numerical problem arise when summing such large quantities. Therefore I normalize all eight features (7 input features and 1 output feature) by subtracting the mean and dividing by the standard deviation of each feature. In this way all features have zero mean and unitary variance. I perform the normalization on the entire data set, before splitting it into a training set and a test set.

After computing the expected utility of each customer, I can rank the customers and establish the customer portfolio. As the customer relationship costs are not provided, I assume that each customer has the same targeting cost. Therefore the marketing budget constraint is equivalent to fix the maximum number of customers in the portfolio.

I test the portfolio selection algorithm 8 as follows. As the goal is to build a customer portfolio that is less risky than a portfolio built using the expected future value as ranking criterion (this corresponds to risk neutrality), I basically need to compare the distributions of two different outcomes, corresponding to two different ranking criteria:

- the total value of a portfolio established by maximizing the expected utility (risk aversion),
- the total value of a portfolio established by maximizing the expected value (risk neutrality).

I use only the test data to provide a realistic estimation of the performance. To find the distributions of the portfolio value I use bootstrap [26]. Bootstrap is a technique that allows us to estimate the distribution of a given parameter as follows. Samples of the same size of the original data set are created by sampling with replacement from the original data set. A sample can therefore contain several copies of the same instance. The parameter is then estimated for each sample created in this way. The different values of the estimated parameter build an empirical distribution which can be used to derive the expected value and variance of the estimated parameter.
I create different sets by sampling with replacement from the test set. Then for each set obtained I build a portfolio that satisfies the budget constraints by using the expected utility ranking criterion or the expected value ranking criterion. The set of all these portfolios gives an estimate of the distribution of the value obtained using a given ranking method. Algorithm 9 summarizes the entire procedure.

**Algorithm 9** Testing the portfolio distribution (bootstrap)

<table>
<thead>
<tr>
<th>Require:</th>
<th>test set $T$, risk-aversion coefficient $c$, number of sampled sets $N$, number of customers in the portfolio (budget constraint) $b$, ranking procedure, i.e. expected utility or expected value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ensure:</td>
<td>empirical distribution $dist$ of the portfolio value using the specified ranking procedure</td>
</tr>
</tbody>
</table>

1. initialize $dist = \{\}$
2. for $i = 1$ to $N$ do
3. extract a sample with replacement from the test set $T$
4. rank according to the specified ranking criterion, use risk-aversion coefficient $c$ in the case of expected utility ranking
5. extract first $b$ customers
6. observe the values realized by the selected customers and sum them to obtain the observed portfolio value $p$
7. $dist = dist \cup \{p\}$
8. end for

The bootstrap procedure is computationally very demanding. I use 200 sampled sets ($N = 200$), the original testing set contains 3695 customers$^3$.

I evaluate customer portfolios using different budget constraints and risk-aversion coefficients. I analyze the following cases: marketing budget allowing us to target 30, 70, and 100 customers; risk-aversion coefficient ($c$) equal to 0 (risk neutrality, or expected value criterion), 0.4, and 0.7. For each combination, I report the following statistics computed from the empirical portfolio distribution provided by the bootstrap algorithm: minimum ($\min$), maximum ($\max$), mean ($\mu$), standard deviation ($\sigma$), difference between the minimum value using expected utility and the minimum value using expected value as ranking criterion ($\delta_{\min}$); this quantity should be positive and shows the risk reduction gained by using utility theory. Moreover, I compute the quantity $\delta_{\min}\%$ as the percentage of increment with respect to the minimum portfolio value if the risk-neutral ($c = 0$) criterion is used. This percentage is important because it tells us whether the difference between the minimum value using risk-neutral and risk-averse criteria is significant or just due to random noise due to applying the bootstrap methods and the estimation of the future expected utility.

$^3$It took 3 hours to compute a distribution using a 2-GHz processor with 768 MB of RAM.
Table 5.4 shows the portfolios in the case that the marketing budget allows us to target at most 30 customers. In this case one can note that the portfolio with the maximum expected value is also the one with the highest risk as measured by the standard deviation ($\sigma$) and by the worst possible outcome ($\min$). Note that if the risk-aversion coefficient is equal to 0.4, the resulting portfolio has a standard deviation reduced by half. Moreover the worst value is 20% above the worst value of the risk-neutral portfolio. By increasing the risk-aversion coefficient to 0.7, the incremental percentage between the two worst values increases to 28%.

The results are different if the budget allows us to target 70 customers, as shown in table 5.5. In this case the percentage of improvement is 6.99% if $c = 0.4$ and 6.25% if $c = 0.7$.

In theory the $\delta_{\min}\%$ should be higher for higher risk-aversion coefficients. However, as the results are obtained by simulation (bootstrap) and by predicting future expected utility using the mixture of Gaussian model, it is reasonable to assume that errors are generated in the process. Moreover, the worst value is obtained by generating 200 bootstrap samples; a higher number of samples would probably generate more consistent results. As proof of the robustness of the approach, note that the standard deviation decreases if the risk aversion increases while the expected value decreases.

The case of a portfolio with 100 customers is illustrated in table 5.6. The standard deviation is reduced by almost 54% if $c = 0.7$.

From the above results, one can note that as the budget increases, and more customers can be targeted, the relative percentage of improvement $\delta_{\min}\%$ of the worst case decreases. The reason for this is that targeting many customers implies that top customers are mixed with low-value customers. As a consequence, the differences between portfolios built using different risk-aversion coefficients are reduced. In fact, even if the ranking is different, the
Chapter 5. Predicting individual value distributions

<table>
<thead>
<tr>
<th>c</th>
<th>min</th>
<th>max</th>
<th>μ</th>
<th>σ</th>
<th>δ min</th>
<th>δ min %</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2209562.5</td>
<td>4973879.3</td>
<td>3348557.6</td>
<td>545461.8</td>
<td>0</td>
<td>0%</td>
</tr>
<tr>
<td>0.4</td>
<td>2226859</td>
<td>3928475.5</td>
<td>2949466</td>
<td>346946.5</td>
<td>17296.5</td>
<td>0.78%</td>
</tr>
<tr>
<td>0.7</td>
<td>2267290.6</td>
<td>3623039.3</td>
<td>2860308.9</td>
<td>251185.5</td>
<td>57728.1</td>
<td>2.6%</td>
</tr>
</tbody>
</table>

Table 5.6: Bootstrap results in the case of a budget targeting 100 customers.

The probability that the same customers are associated with two different portfolios increases as the number of customers increases. In the extreme case, if all customers are targeted, the ranking method has no influence on the final portfolio which includes the entire customer base.

5.8 Conclusions and managerial implications

Using Gaussian mixtures one can predict the future value distribution of each customer. This allows us to explicitly consider the uncertainty in the decision-making process and to apply financial insights to optimize the value-risk tradeoff when building customer portfolios.

As shown in the case study, several issues can be addressed once the value distribution is known. Defectors can be detected, loyal or high potential customers can be identified, the value of each customer can be forecasted, and a portfolio of customers can be created according to financial investment criteria considering the value-risk tradeoff and the risk profile of the decision maker. Moreover, the Gaussian mixture model can be used to segment customers according to the hidden variables (i.e. the latent classes) defining the mixture. In fact, probability density estimation using Gaussian mixtures is deeply related to probabilistic clustering using the EM algorithm [88]. Figure 5.5 gives an overview of the marketing applications of Gaussian mixtures.

Figure 5.5: Modeling future customer value using a mixture of Gaussians.
The allocation of marketing resources from a financial perspective has recently been identified as a major research direction by the Marketing Science Institute [39]. This chapter contributes to the field by providing analytical tools and models that allow us to manage a customer portfolio as a financial portfolio by considering the value and the risk involved in the future prediction. Customers are seen as financial assets, which contribute to the total value generated by a company. As in any financial asset management process, knowledge of the value-risk profile is fundamental for the rational allocation of resources [27]. From this perspective, marketing managers should carefully select the profiles of the customers they want to target.

As shown in the case study, different customers portfolios can be created using appropriate customer-selection criteria. A risk-neutral criterion maximizes the expected value of the future customer portfolio; but the downside risk is not minimized. On the other hand, a risk-averse criterion leads to portfolios with a lower future expected value but also with a lower variance. It is interesting to note how the worst value improves if a risk-averse utility function is used to build the customer portfolio. For instance, in the case of a portfolio of 30 customers, the worst case of the risk-averse portfolio (using a risk tolerance parameter of 0.7) is much better than the worst case of a portfolio obtained by maximizing the future expected value: the difference of the two worst cases is larger than 172000 €, which is quite high given the small number of customers. A risk-sensitive resource allocation strategy would therefore have a relevant impact for the decision maker.

Is it better to invest in customers whose value is quite stable, i.e. low variance, but not so high, or is it more appropriate to invest in customers whose future value is not so certain, i.e. high variance, but can be very high? The answer to this question depends on the risk profile of the decision maker. If the marketing strategy is very aggressive, for instance because the company is entering a new market or launching a new product, then the marketing manager can exhibit a risk-seeking behavior. In this case resources will be allocated to customers that have a very uncertain future value. The uncertainty can be simply due to the fact that the company’s database does not yet contain many customers which renders the estimation of their future value contribution difficult.

Uncertainty can also be incorporated in the Gaussian mixture model by imposing constraints on the covariance matrix of the mixture components [4]. These constraints represent a kind of prior knowledge. Uncertainty can be considered as well at the feature level. For instance, if the frequency in the last six and twelve months is not known for a given customer, one can initialize these frequency values to the average observed in the data, i.e. freq6 would be set to the average of the known freq6 values. This is a well known approach for dealing with missing values [88]. In the context of customer relationship management, this approach addresses the issue of analyzing new customers for which not much historical data is available. Alternatively, the marginal distribution of the known input features can eas-
ily be derived as described in section 5.4, and the unknown input features are therefore not considered in the conditional density predicting the output, i.e. the customer value.

The principle of expected utility [62] nicely adapts to the mixture of Gaussian distributions. In fact, a closed form for the expected utility can be derived that allows us to extend and generalize the traditional “rank and cut” practice used to build customer lists for the allocation of marketing resources [63]. It extends the common practice, which considers only the expected future value, by considering uncertainty. It generalizes the common practice because the expected utility principle leads to the maximization of the expected value in the case of risk neutrality.

Finally, as shown in the case study, the predictive performance of the Gaussian mixture model is in line with that of other well-known state-of-the art predictive algorithms. Compared with other algorithms, the mixture of Gaussians approach has the advantage of estimating as much information as possible by providing the entire distribution of future outcomes and not just a numeric value (regression) or a class (classification).
Chapter 6

Conclusions

6.1 Summary

In this thesis, I discuss how to allocate marketing resources in order to optimize customer equity by developing analytical models which link marketing policies to customer lifetime value and which estimate the financial profile of customers. The thesis addresses the following three issues: a) maximization of customer lifetime value by finding the optimal marketing policy tailored to the individual customer, b) evaluation of customers’ financial profiles, and c) risk-sensitive allocation of marketing resources by establishing a portfolio of customers.

As described in chapters 1 and 2, these issues are not addressed by the current models. In fact, most of the lifetime value models described in the literature do not address heterogeneity in customer lifetime value. Moreover the issue of maximizing customer lifetime value at the individual or segment level is not explicitly addressed. In fact, customer lifetime value is seen as an \textit{exogenous} variable, i.e. not depending on the marketing action. Markov chain models can deal with individual lifetime value estimation, but the estimation of the underlying chain (i.e. states, transaction probabilities, rewards) has not been much addressed in the literature. Current approaches use parametric regression techniques [78], [21] to predict the transition probabilities, but the states are chosen \textit{ad hoc} and the expected rewards do not depend on the marketing actions. Parametric regression methods to predict transition probabilities can lead to a high bias and consequently to wrong marketing recommendations [82]. The main issue with Markov chains is that these models measure lifetime value but do not explicitly maximize it by finding the best marketing actions at an individual customer level. Moreover, currently there is no model which deals with risk by providing information about the volatility of the predicted lifetime values. Consequently, risk is not considered when allocating marketing resources.

The maximization of customer lifetime value and the evaluation of customers’ financial
profiles are addressed in chapter 3. I develop a methodology, based on cross-validation and customer segmentation algorithms, for the estimation of robust Markov Decision Processes modeling the dynamic relationship between the customers and the company. I use a non-parametric approach to estimate the model. Using dynamic programming, the marketing policy (i.e. a mapping from customer states to marketing actions) which maximizes the expected customer lifetime value is found. Customer heterogeneity across states (i.e. customer segments) and marketing action heterogeneity is addressed by tailoring marketing actions to individual customers. To evaluate the financial profile of customers, i.e. the lifetime value distributions, I use Monte Carlo simulation coupled with bootstrap. In this way both the uncertainty in the model parameters and the uncertainty in the customer behavior is captured.

Once the financial profile of customers has been estimated, risk-sensitive allocation of marketing resources is addressed, in chapter 4. I discuss how to allocate a limited marketing budget by formulating and solving the associated constrained optimization problem. The objective is to maximize expected value while minimizing risk. I show that unlike assets exchanged in financial markets, when evaluating the risk profile of customer assets there is not always an effective value-risk tradeoff. In fact, as shown using some examples, a risk-averse decision maker might prefer a high-value and high-risk customer segment over a customer segment with lower expected value and lower risk. This is the case when the value distributions of the two segments do not overlap and the worst case of the risky segment is still better than the best case of the less risky segment. I discuss the conditions under which a value-risk tradeoff is a relevant issue. For instance, the uncertainty due to the model parameters can have a large impact on the tradeoff. For this reason it is important to consider the parameter uncertainty when estimating the financial profiles of customers, as done in chapter 3.

While in chapters 3 and 4 customer lifetime value is estimated and maximized at a customer segment level, in chapter 5 I develop an alternative approach for the establishment of a customer portfolio. I use mixtures of Gaussians to predict the lifetime value distribution of each customer. While addressing lifetime value heterogeneity across customers, this approach does not maximize lifetime value. In fact, customer portfolio optimization is achieved through selection. As shown in a case study, several issues can be addressed once the value distribution is known. Defectors can be detected, loyal or high potential customers can be identified, the value of each customer can be forecasted, and a portfolio of customers can be created according to financial investment criteria considering the value-risk tradeoff and the risk profile of the decision maker. Different customer portfolios can be created using appropriate customer selection criteria. A risk-neutral criterion maximizes the expected value of the customer portfolio, but the downside risk is not minimized. On the other hand, a risk-averse criterion leads to portfolios with a lower future expected value but with a lower downside as well. The principle of expected utility nicely adapts to mixtures of Gaussians. I
derive a closed form for predicting the expected utility of each customer. This allows us to extend and generalize the traditional “rank and cut” practice used to build customer lists for the allocation of marketing resources. The developed approach extends the common practice, which considers only the expected future value, by considering uncertainty. It generalizes the common practice because the expected utility principle leads to the maximization of the expected value in the case of risk neutrality.

To summarize, the main contributions of this thesis are:

- the estimation of robust Markov Decision Processes maximizing the predictive accuracy of the model and consequent maximization of expected lifetime value,
- the estimation of customers’ financial profiles and expected utilities using simulation and mixture distributions,
- the use of the mean-variance Markowitz framework and utility theory for the establishment of customer portfolios optimizing the value-risk tradeoff.

The allocation of marketing resources from a financial perspective has recently been identified as a major research direction by the Marketing Science Institute [39]. This work contributes to the field by providing analytical tools and models for efficient allocation of marketing resources by focusing on the financial impacts of marketing investments and on the optimization of customer equity.

### 6.2 Managerial implications

The managerial implications of this thesis are manifold. Firstly, the understanding of customer dynamics enables evaluation of the impact of a sequence of marketing actions in time. Secondly, marketing managers can fix the planning horizon and find the best marketing policy optimizing the financial value at an individual customer level. In fact, once the optimal policy has been determined, it is possible to associate to each customer in the database a marketing action which depends on the customer features (e.g. age, sex, nationality, area of residence, etc.) and on the customer’s past experience (e.g. number of purchases, number of marketing campaigns received so far, number of responses to past campaigns, etc.). The questions of whom to target, how, and when are therefore answered.

*Sequential patterns*, such as first sending a campaign which offers flight points and, in the case of positive response, sending subsequently a campaign offering flight tickets in exchange for accumulated points and money, are automatically detected and leveraged. Other sequential patterns include the prevention of saturating customers by repeatedly targeting them with campaigns, and the sending of tactical campaigns (e.g. newsletters) for
the purpose of strengthening brand awareness (without expecting any immediate return or response). These kinds of campaigns should be sent only to those customers for which they stimulate future sales, and not indistinctively to the entire customer base. Also, focusing on customers who generated much value in the past, but are not likely to continue with their buying pattern in the future, is not an efficient marketing strategy.

Sequential patterns can also imply up-selling and cross-selling. For instance, customers who responded to a campaign for using a high-speed internet connection (e.g. ADSL), can decide later to pay a premium price for having more capacity (i.e. up-selling), or for using additional services such as music downloads (i.e. cross-selling), if appropriately targeted by subsequent marketing campaigns. Moreover, by estimating the impact that marketing actions have on the future value distributions, it is possible to test hypotheses. For instance, the hypothesis that offering customers a loyalty program will stimulate cross-selling can be rejected.

As the customer life-cycle can be modeled with Markov Decision Processes, the sequence of marketing actions stimulating cross-selling and up-selling can be found by using dynamic programming algorithms. In other words, sequential patterns are detected and exploited for marketing purposes. From a data-mining perspective, the hidden information stored in the company’s database is discovered and exploited. Using information technology, the execution of the marketing policy can be fully automated.

Thirdly, marketing managers can maximize the creation of financial value by distributing their budgets both across different marketing activities and across customers. Moreover, the return on marketing investment can be forecasted and compared with the observed return at the end of the planned investment horizon. This allows marketing managers to refine the model in the future\(^1\).

One of the reasons why advanced predictive models are not commonly used by companies for customer relationship management is their inaccuracy. In the most general case, a highly accurate prediction of the future lifetime value will never be possible in a non-contractual setting, because customers are not bound to the company and are free to change their buying patterns. By considering the uncertainty in the future value predictions, i.e. the financial profile of customers, managers can reduce risk by diversification. The quantification of the risk involved in the customer relationship, using either Markov Decision Processes or Gaussian mixtures, can help decision makers to select portfolios yielding an optimal value-risk tradeoff. For example, marketing activities can be addressed to different

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\(^1\)Iterative estimation of Markov Decision Processes might be necessary to improve the predictive accuracy if the optimal marketing policy differs significantly from the historical policy. In fact, one requirement of Markov Decision Processes estimated using a non-parametric approach is that some examples of the application of the optimal policy are observed in the historical data. A gradual change in the marketing policy can lead to a more stable estimation of the customer dynamics.
groups of customers: a) loyal customers buying the company’s products regularly, b) new customers acquired by entering a new market with a new product, c) sporadic customers which buy irregularly, etc. Each of these groups can exhibit a different value-risk profile. According to the risk appetite of the decision maker, the available marketing budget can be allocated so as to optimize the value-risk tradeoff of the projected return on investment.

6.3 Future research

Future research can include the application of the developed models to other industries and companies. Once customer dynamics are modeled and optimal marketing policies are found, it would be interesting to implement field tests to see how good the optimal policy performs in reality, compared to the current policy of the company. Such tests typically consider two randomly chosen groups of customers: the treatment group and the control group. Customers in the treatment group are targeted with the optimal marketing policy whereas customers in the control group are targeted with the historical marketing policy. After a given time period, the performance of the two policies can be observed from the collected data and the hypothesis that the optimal policy outperforms the historical one can be tested.

During a field test, the iterative process leading to the Markov Decision Process can be analyzed. The steps of the process are the following: a) estimate the Markov Decision Process modeling customer dynamics using transactional data, b) find the optimal marketing policy using dynamic programming, c) test the optimal marketing policy and collect new transactional data, d) repeat the procedure. This iteration leads to a sequence of models. These models progressively capture the dynamics induced by marketing policies which might not be observed in the original transactional data.

Another future research direction includes the analysis of the circumstances in which a value-risk tradeoff takes place. The lifetime value distributions depend on the particular customers and company. An experimental analysis, including, for instance, companies belonging to different industries, could reveal situations in which a risk-sensitive allocation of marketing resources is more pertinent.

The proposed model can be extended in order to explicitly consider the growth of the customer base and thus better link the customer equity to the market value of the company. For instance, by introducing a fictitious state modeling the entrance of new customers at each step in time, the long-term revenues and costs generated by new customers can be considered. In addition to applying dynamic programming to the estimated Markov Decision Process, other marketing applications can be investigated. For example, the probability that a customer reaches a particular state (e.g. defection state) given his current features can be estimated, and actions avoiding to reach that state can be found. Moreover, the most
likely future path can be predicted. The evolution of the customer equity and the required marketing investments can be forecasted regularly (e.g. each month). This would be relevant for corporate financial planning.

A customer equity research consortium has been established for the purpose of developing these and other research activities in the field. This consortium includes leading companies in the financial services, telecommunication, information technology, transportation, and the automotive industries.
Appendix A

Optimization under uncertainty

A.1 Introduction

In this appendix I briefly review risk management theory and utility theory. The purpose is to introduce the financial portfolio optimization techniques used in chapters 4 and 5 in order to efficiently allocate the marketing budget.

A deterministic optimization problem can be stated as follows:

\[
\min_{x} f(x) \quad \text{subject to} \quad x \in F \subseteq \mathbb{R}^n.
\]  

(A.1)

The objective function to minimize is \( f(x) : \mathbb{R}^n \to \mathbb{R} \) and the constraints restrict the solution to belong to the set \( F \) (feasibility set). If the objective function and the feasibility set are convex, there are efficient numerical methods (e.g. gradient descent search) to find the \textit{global} minimum. If the objective function or the feasibility set are not convex, numerical methods are not guaranteed to find a global minimum, because there could be many \textit{local} optima. In this case optimization heuristics such as simulated annealing [44], threshold acceptance [30], or genetic algorithms [31], [58] can be used. These methods perform a random search in the solution space and are not guaranteed to find a global optimum, however they are more robust and perform better in many practical situations than traditional deterministic gradient descent methods [31]. If the objective function is stochastic (i.e. non-deterministic) the above problem can be formulated as follows:

\[
\min_{x} f(x, y) \quad \text{subject to} \quad x \in F \subseteq \mathbb{R}^n, \quad y \in \mathbb{R}^m.
\]  

(A.2)
The objective function to minimize is \( f(x, y) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \). The first argument, \( x \), is the (deterministic) decision variable (e.g. weights of the assets building a portfolio) to be determined by optimizing the objective function. The second argument, \( y \), is a random variable (or random vector) with a given probability density function \( p_y \) (e.g. distributions of the returns on investment of several assets). Since \( y \) is random, for every particular \( x \in \mathbb{R} \), the function \( f(x, y) \) is a random variable with an associated univariate probability density function which can be derived from the distribution of \( y \). In order to solve the stochastic optimization problem, one needs to define a mapping from the probability distribution of a random variable to a real value. Once this is accomplished, problem (A.2) can be reformulated as problem (A.1). I consider two approaches for the establishment of such a mapping: a) financial risk management techniques, and b) utility theory. These two approaches are not mutually exclusive, indeed they are related, but their underlaying theoretical motivations come from different areas. Risk management is more concerned with describing risk as the deviation from a target (e.g. expected gain) and the axiomatization of risk measures. Utility theory derives from an axiomatization of the behavior of a rational investor, which implies the modeling of choices under risk (i.e. uncertainty).

A.2 Risk management

The risk management approach defines a loss function, representing the stochastic nature of the possible losses resulting from an investment decision, and a risk measure evaluating the loss function. Once a risk measure has been defined, risk can be reduced by diversification, that is by aggregating different investments to reduce the overall risk of the resulting portfolio. A loss function can be defined as follows:

\[
l_y(x) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \\
x \in F \subseteq \mathbb{R}^n \\
y \in \mathbb{R}^m.
\]  

The vector \( x \) represents the investment decision variables while \( y \) is a random vector representing the risk factors which cannot be influenced by the decision maker. The decision vector defines, for instance, a portfolio of assets. The set \( F \) represents all the investment options (e.g. possible portfolios). The risk factors model uncertainties that influence the possible loss of an investment, such as market parameters in the case of a portfolio of equities. Note that for a given investment decision vector, the loss function is a random variable with an associated probability density function. The loss function is equal to the negative gain function, therefore the minimization of the loss function corresponds to the maximization
of the gain (or profit) function. In the case of a portfolio whose value is given by a linear combination of decision variables and risk factors, the loss function is:

\[ l_y(x) = -\sum_{i=1}^{n} x_i \cdot y_i = -x^T y, \tag{A.4} \]

where bold notation is used to represent vectors. The decision variable \( x_i \) and the random variable \( y_i \) represent, respectively, the quantity and the unit-value of asset \( i \). The minimization of the risk can therefore be reformulated as the minimization of a risk measure (e.g. variance), which depends on the probability density function of the loss function subject to some constraints (e.g. expected loss is less than a given threshold value).

There are several definitions and categorizations of risk measures in the literature [2], [64]. I define a risk measure according to [68] as a function of the probability density function of the loss (or the gain). In other words, a risk measure is a mapping from a univariate probability density function to a real value. Risk measures are characterized by different properties. Several authors describe which properties good risk measures should have (e.g. [3], [1]). For the scope of the analysis, and for the sake of simplicity, I do not describe all these properties in detail. Rather, I provide an overview of the most used risk measures in practice, and of their related properties.

### A.2.1 Variance

For historical reasons, the variance and the standard deviation (i.e. the positive square root of the variance) of the loss (or gain) distribution are the most known risk measures in finance. The standard deviation measures the average spread from the expected loss (or expected gain). The popularity of these risk measures is due to the mean-variance model introduced by Markowitz [56] in 1952, which prescribes the selection of those portfolios with the minimum expected loss (i.e. the maximum expected gain) for a given level of risk, as measured by the standard deviation, or equivalently by the variance. As both the variance and the expected value are convex functions, finding the optimal portfolio is a convex optimization problem for which efficient solution methods exist [55]. For different levels of risk, different portfolios are obtained. The set of these optimal portfolios is called efficient frontier. Alternatively, the efficient portfolios can be found by minimizing the risk for a given level of expected loss. Assuming a linear loss function as defined in equation (A.4), the expected loss is:

\[ \mu = \mathbb{E}[l_y(x)] = -x^T \mu_y, \]

where \( \mu_y \) is the expected value of the random vector \( y \). The variance of the portfolio’s loss function is given by:

\[ \sigma^2 = \text{var}[l_y(x)] = x^T C x, \]
where $C$ is the covariance matrix of the random vector $y$ and the element $(i, j)$ is the covariance of the random variables $y_i, y_j$:

$$C(i, j) = cov(y_i, y_j) = \sigma_{y_i, y_j}.$$ 

To show how risk can be reduced through diversification I consider a portfolio with only two assets, in this case:

$$\sigma^2 = x_1^2 \sigma_{y_1}^2 + x_2^2 \sigma_{y_2}^2 + 2x_1 x_2 \rho_{y_1, y_2} \sigma_{y_1} \sigma_{y_2}.$$ 

The correlation coefficient of two random variables is defined as:

$$\rho_{y_1, y_2} = \frac{\sigma_{y_1, y_2}}{\sigma_{y_1} \sigma_{y_2}}.$$ 

The correlation coefficient ranges from -1 to +1 [60], the standard deviation can be reformulated in terms of the correlation coefficient:

$$\sigma^2 = x_1^2 \sigma_{y_1}^2 + x_2^2 \sigma_{y_2}^2 + 2x_1 x_2 \rho_{y_1, y_2} \sigma_{y_1} \sigma_{y_2}.$$ 

The maximum variance is obtained when $\rho_{y_1, y_2} = 1$, and is equal to:

$$\sigma_{\text{max}}^2 = x_1^2 \sigma_{y_1}^2 + x_2^2 \sigma_{y_2}^2 + 2x_1 x_2 \sigma_{y_1} \sigma_{y_2} = (x_1 \sigma_{y_1} + x_2 \sigma_{y_2})^2$$ 

therefore:

$$\sigma \leq x_1 \sigma_{y_1} + x_2 \sigma_{y_2}.$$ 

The standard deviation of the portfolio is therefore always less or equal to the sum of the standard deviations of the portfolio’s assets. If the assets are perfectly correlated, then the sum is equal to the portfolio’s standard deviation. This examples shows that as long as the correlation coefficient is less than one, the risk is reduced by diversification when two or more assets are combined into a portfolio. The extreme case, when the correlation coefficient is equal to -1, a portfolio with zero variance can be obtained, in this case there would be no uncertainty in the future loss or gain of an investment. In practice, assets which are perfectly uncorrelated (i.e. their correlation coefficient is equal to -1) are not easy to find. In a similar way, it is possible to show that the variance cannot increase through diversification [56].

This example shows as well that the standard deviation is a sub-additive risk measure [1], [2]. This means that given two assets $A$ and $B$, and a portfolio composed of these two assets, the standard deviation of the portfolio is less than or equal to the sum of the standard deviations of the two assets, formally:

$$\sigma_{A+B} \leq \sigma_A + \sigma_B.$$ 

The sub-additive property is probably the most relevant property of a risk measure because it encodes the principle of risk diversification. Not all commonly used risk measures obey to this fundamental property.
As next example, let us consider the case of a portfolio built of \( N \) assets, all of which have the same expected value and the same variance \( \sigma^2 \), but are uncorrelated (i.e. \( \rho = 0 \)). Moreover, let us assume that the same amount of investment is allocated to each asset. This means that:

\[
x_i = \frac{\beta}{N}, \quad \forall i \in [1, \cdots, N],
\]

where \( \beta \) is the total available budget and \( x_i \) is the amount of investment allocated to asset \( i \).

The variance of the portfolio’s loss distribution is in this case:

\[
\sigma^2_N = \sum_{i=1}^{N} x_i^2 \text{var}(x_i) = \sum_{i=1}^{N} \frac{\beta^2}{N^2} \sigma^2 = \frac{\beta^2}{N} \sigma^2.
\]

From the above formula it is easy to see that the portfolio variance decreases with the number of assets, in the limit case:

\[
\lim_{N \to \infty} \sigma^2_N = 0.
\]

The optimal decision vector \( x^* \) of an efficient portfolio is the solution of the following convex optimization problem [56]:

\[
\begin{align*}
\min_x & \quad -x^T \mu_y \\
& \quad x^T C x \leq \tilde{\sigma}^2 \\
& \quad q^T x = \beta
\end{align*}
\]  

(A.5)

where \( q \) is the asset cost vector and \( \beta \) is the total available investment budget. The minimum acceptable variance is \( \tilde{\sigma}^2 \) which is a parameter of the problem and defines the position of the portfolio in the efficient frontier. An alternative formulation is the following optimization problem:

\[
\begin{align*}
\min_x & \quad x^T C x \\
& \quad -x^T \mu_y \leq \tilde{\mu} \\
& \quad q^T x = \beta
\end{align*}
\]  

(A.6)

where \( \tilde{\mu} \) is the maximum acceptable expected loss. Finally, another alternative formulation is the following:

\[
\begin{align*}
\min_x & \quad -x^T \mu_y + r \cdot x^T C x \\
& \quad q^T x = \beta \\
& \quad r \in \mathbb{R}^+
\end{align*}
\]  

(A.7)

where \( r \geq 0 \) is the risk-aversion coefficient which defines the position of the portfolio in the efficient frontier. All these three problems are convex optimization problems, and have the
same efficient frontier [48]. In fact, problem (A.7) is the Lagrangian form of the other two optimization problems [55].

Another formulation of the efficient portfolio, used by several authors [54], is to minimize the variance (i.e. the risk) for a given level of expected loss. According to this definition, the efficient portfolio is obtained by solving the following optimization problem:

\[
\begin{align*}
\min_x & \quad x^T C x \\
& \quad -x^T \mu_y = \hat{\mu} \\
& \quad q^T x = \beta.
\end{align*}
\]  
\tag{A.8}

The above problem differs from problem (A.6) only in the equality constraint on the expected loss. It is possible to prove that these two problems have the same optimal solution and are therefore equivalent; the formulation (A.6) is more used in practice because it is easier to solve computationally. Problem (A.8) is a convex optimization problem with only equality constraints, if the matrix \(C\) is positive definite and the matrix \((\mu_y q)^T\) is invertible, then this problem can be solved in closed form [55].

As measures of risk, variance and standard deviation have a main drawback: they measure the spread around a target value (i.e. the expected loss) but do not make distinction between positive and negative deviation from the target. Variance is therefore a good risk measure only if the loss distribution is symmetric around its mean. In many practical cases, however, loss distributions are not symmetric. Therefore, the use of the mean-variance model is not well suited. In the next section, I introduce other risk measures which deal with asymmetric distributions by focusing on the downside risk, i.e. the upper tail of the loss distribution.

A.2.2 Value-at-Risk (VaR)

The Value-at-Risk (VaR) [42] is a widely used risk measure today. Given some confidence level \(\alpha \in (0, 1)\), the Value-at-Risk of a portfolio with loss \(L\) is given by the smallest number \(l\) such that the probability that the loss exceeds \(l\) is less than \((1 - \alpha)\). Formally:

\[
VaR_{\alpha}(L) = \inf\{l \in \mathbb{R} : P(L > l) \leq 1 - \alpha\} = \inf\{l \in \mathbb{R} : F_L(l) \geq \alpha\}
\]  
\tag{A.9}

where \(F_L(\cdot)\) is the cumulative density function of the random variable \(L\). The VaR of a portfolio is therefore the \(\alpha\)-quantile of the loss distribution. If the gain distribution \(G = -L\) is used instead of the loss distribution, the following relationship holds:

\[
VaR_{\alpha}(L) = -VaR_{1-\alpha}(G).
\]
Given a loss function \( l_y(x) \), where \( x \) are the decision variables and \( y \) are the stochastic risk factors, the Value-at-Risk optimization problem can assume one of the following forms (the constraint on the budget is omitted):

\[
\begin{align*}
\min_x \ & VaR_\alpha(l_y(x)) \\
\mathbb{E}[l_y(x)] & \leq \hat{\mu}
\end{align*}
\]  

(A.10)

or,

\[
\begin{align*}
\min_x \ & \mathbb{E}[l_y(x)] \\
& VaR_\alpha(l_y(x)) \leq \hat{\ell}.
\end{align*}
\]  

(A.11)

In problem (A.10), the Value-at-Risk is minimized by imposing that the expected loss is equal or less than a given threshold value. In problem (A.11) the expected loss is minimized given a threshold on the Value-at-Risk. As the Value-at-Risk is in general a non-convex function [69] which can have several local minima, both problems are non-convex optimization problems. For these types of problems, numerical optimization procedures based on deterministic algorithms tend to fail. Optimization heuristics such as genetic algorithms or threshold acceptance have been successfully used in these cases [30]. One drawback of Value-at-Risk is that this risk measure does not satisfy the sub-additivity property, which is required by many risk measurement systems. For this reason VaR is said to be a non-coherent risk measure [3]. This implies that risk is not always reduced by diversification if it is measured using the Value-at-Risk. Another drawback of this risk measure is that it does not take into consideration the cases which are worst than the VaR, this could lead to an underestimation of risks in the case of very big, but unlikely, losses, i.e. extreme events.

**A.2.3 Conditional Value-at-Risk (CVaR)**

Conditional Value-at-Risk (CVaR), or Expected Shortfall (ES), overcomes some drawbacks of VaR. Given some confidence level \( \alpha \in (0, 1) \), the \( CVaR_\alpha \) of a portfolio with a loss distribution represented by a continuous cumulative density function \( F_L(\cdot) \) is defined as the conditional expectation of \( L \) given that the loss exceeds the \( VaR_\alpha \). Formally:

\[
CVaR_\alpha(L) = \mathbb{E}[L|L \geq VaR_\alpha(L)].
\]  

(A.12)

The above definition is valid in the case that the cumulative density function of the loss is continuous. If this is not the case, CVaR can be defined as the solution of the following optimization problem [48], [69], [87]:

\[
CVaR_\alpha(L) = \inf_{\nu} \left\{ \nu + \frac{1}{1 - \alpha} \mathbb{E}[L - \nu]^+ : \nu \in \mathbb{R} \right\}
\]  

(A.13)
where the function \( [x]^+ = \max(x, 0) \). Definition (A.13) is more general than the definition (A.12), it can be shown \([87]\) that: a) if the cumulative density function of the loss is continuous, then the two definitions are equivalent, b) the optimal \( \nu \) is equal to \( \text{VaR}_\alpha \), c) if the loss function is convex, then CVaR is convex. The Conditional Value-at-Risk is a coherent measure of risk \([69]\), in particular, it obeys to the sub-additive property. As in the case of the VaR, given a loss function \( l_y(x) \), where \( x \) are the decision variables and \( y \) are the stochastic risk factors, the CVaR optimization problem can assume one of the following forms (the constraint on the budget is omitted):

\[
\min_x \ CVaR_\alpha(l_y(x)) \quad \text{s.t.} \quad \mathbb{E}[l_y(x)] \leq \hat{\mu} \tag{A.14}
\]

or,

\[
\min_x \ \mathbb{E}[l_y(x)] \quad \text{s.t.} \quad CVaR_\alpha(l_y(x)) \leq \hat{l}. \tag{A.15}
\]

Both (A.14) and (A.15) are convex optimization problems if the loss function is convex. By exploiting definition (A.13), the minimization of the Conditional Value-at-Risk can be reformulated as follows:

\[
\min_x \ CVaR_\alpha(l_y(x)) = \min_{x, \nu} \ F_\alpha(x, \nu)
\]

where

\[
F_\alpha(x, \nu) = \nu + \frac{1}{1 - \alpha} \mathbb{E}[l_y(x) - \nu]^+.
\]

The expected value in the above formula can be approximated by sampling from the probability density function of the underlying risk factors \( f_y \):

\[
\tilde{F}_\alpha(x, \nu) = \nu + \frac{1}{(1 - \alpha)n} \sum_{k=1}^{n} [l_{y, k}(x) - \nu]^+.
\]

Given \( n \) samples, which can be generated either by Monte Carlo or historical simulation, and the auxiliary variables \( z_k \), the minimization of the CVaR can be reduced to the following linear programm (LP) if the loss function \( l_y \) is linear in \( x \):

\[
\min_{x, \nu, z_k} \nu + \frac{1}{(1 - \alpha)n} \sum_{k=1}^{n} z_k
\]

\[
z_k \geq l_{y, k}(x) - \nu \tag{A.16}
\]

\[
z_k \geq 0
\]

\[
\forall k \in [1, n].
\]

Additional linear constraints, e.g. \( \mathbb{E}[l_y(x)] \leq \hat{\mu} \), can be added to the above optimization problem. Constraints on the CVaR can be expressed with linear equations \([48]\), therefore...
both problems (A.14) and (A.15) can be reduced to linear programs if the loss function is linear. There are very efficient algorithms to solve linear optimization problems or convex optimization problems. Therefore Conditional Value-at-Risk optimization problems are, in general, easier to solve than VaR optimization problems. This is the main advantage of using CVaR as a risk measure. Another advantage of CVaR over VaR is that the former takes into consideration possible losses which exceed the Value-at-Risk. This leads to a more informed risk measure taking into account the entire range of worst-case losses and not only the VaR. Moreover, since the CVaR is always greater or equal to the VaR, minimizing the CVaR of a given portfolio gives an upper limit to the Value-at-Risk of that portfolio, which is equal to the optimal $\nu$ of problem (A.16).

A.3 Utility theory

Utility theory addresses the problem of choice under uncertainty and provides quantitative methods for analyzing decisions based on the axioms of consistent choice [62]. The general decision problem can be stated as follows. A decision maker has to choose an alternative belonging to a set of alternatives (opportunity set) $A_1, \ldots, A_n$, each of which has a stochastic outcome (i.e. consequence). For each alternative, the probabilities of the possible outcomes are known.

If an appropriate utility is assigned to each possible outcome, then alternatives can be ranked according to their expected utility. Different sets of axioms modeling the behavior of a rational decision maker have been formulated so far [62], [80], [52], and [70]. They all agree on selecting the alternative with the highest expected utility. In general, utility functions are subjective and reflect the decision maker’s attitude toward risk.

Let us now analyze a simple example to motivate the need of a utility function modeling the preferences of the decision maker and his tolerance to risk (e.g. risk neutral, risk averse, risk seeking). Suppose an investor has to choose between the following two alternatives:

1. $A_1$: play a gamble with equal probabilities to win 100 € or to lose 20 €, and

Different investors can take different decisions according to their attitude towards risk. The average outcome of the alternative $A_1$ is:

$$\frac{1}{2} \cdot 100 - \frac{1}{2} \cdot 20 = 40.$$ 

If the investor is risk neutral, he will be indifferent between selling alternative $A_1$ for 40 € and choosing it. A risk-neutral investor would prefer to play the gamble $A_1$ rather than to
obtain with certainty 30 € and not be able to play the gamble. In fact, the value of $A_1$ is 40 € which is more than 30 €. On the other hand, a risk-averse investor might prefer 30 € for certain instead of taking the risk involved in alternative $A_1$. In other words, for a risk-averse investor, the potential utility of winning 100 € does not compensate for the pain of a potential loss of 20 €, therefore he will not play the gamble.

To summarize, for a risk-neutral investor the value of the gamble is equal to its expected outcome (e.g., 40 € in alternative $A_1$). For a risk-averse investor the value is less than the expected outcome. For a risk seeking investor, the value of a gamble is more than its expected outcome.

The fact that for a risk-neutral attitude the value of a gamble is equal to its expected value can be motivated through the “Weak Law of Large Numbers” [45]. According to this law, the average outcome ($\bar{x} = \frac{1}{n} \sum x_i$) for a large number of independent random variables $x_i$ (outcomes of the alternatives) with $E[x_i] = \mu$, stochastically converges to the expected outcome $\mu$ of the different alternatives. Therefore, over many repeated decisions, the average outcome will be very close to the mean $\mu$. In this case, a risk seeking decision maker who is willing to evaluate (and buy) the alternatives at more than their expected value, would lose money in the long term.

Risk aversion is motivated by the following two remarks:

- If a decision can be taken only once, then the outcome is not guaranteed to be close to the expected value of the selected alternative.

- An alternative with a high expected value (related to the other alternatives) can have a very negative outcome for the decision maker. Even if the probability of this outcome is small, the decision maker might prefer to select another alternative with a lower expected value but not a such negative possible outcome.

A risk-averse investor would value alternative $A_1$ less that its average outcome which is:

$$E[x|A_1] = \sum_{i=1}^{m} x_i P(x_i),$$

where $x_i$ are the $m$ potential outcomes of alternative $A_1$ and $P(x_i)$ are their probabilities.

The risk-averse investor would evaluate the gamble less than its expected value according to his utility function:

$$E[u(x)|A_1] = \sum_{i=1}^{m} u(x_i) P(x_i),$$

here the values $x_i$ are substituted with their utility. In this way, it is possible to assign a level of “happiness” to each outcome and penalize extremely averse outcomes.
A.3. Expected Utility Theorem

The Expected Utility Theorem states that the alternative with the highest expected utility should be chosen by the decision maker. The theorem is based on the following axioms of consistent choice. I denote with $c_i$ a possible outcome (i.e. consequence) of a decision. The axioms and the proof of the theorem are based on [45], a similar formulation can be found in [27].

1. **Transitivity**: if $c_i$ is preferred to $c_j$, and $c_j$ is preferred to $c_k$, then $c_i$ is preferred to $c_k$.

2. **Reduction**: if the standard rules of probability can be used to show that two alternatives have the same probability for each $c_i$, then the two alternatives are equally preferred.

3. **Continuity**: if $c_i$ is preferred to $c_j$ and $c_j$ is preferred to $c_k$, then there is a $p$ ($0 \leq p \leq 1$) such that an alternative with a probability $p$ of yielding $c_i$ and a probability of $1 - p$ of yielding $c_k$ is equally preferred to $c_j$.

4. **Substitution**: if two consequences are equally preferred, then one can be substituted for the other in any decision without changing the preference ordering of alternatives.

5. **Monotonicity**: for two alternatives which each yield either $c_i$ or $c_j$, where $c_i$ is preferred to $c_j$, then the first alternative is preferred to the second if it has a higher probability of yielding $c_i$.

If an investor acts in accordance to these axioms, then the investor’s behavior is indistinguishable from one who makes decisions based on the following Expected Utility Theorem.

**Theorem A.3.1 (Expected Utility).** If the axioms of consistent choice hold, then there exists a function $u(c_i)$ such that alternative $A$ is preferred to alternative $B$ if and only if:

$$\sum_{i=1}^{n} p(c_i|A)u(c_i) > \sum_{i=1}^{n} p(c_i|B)u(c_i),$$

where $p(c_i|A)$ is the probability of $c_i$ if $A$ is selected and $p(c_i|B)$ is the probability of $c_i$ if $B$ is selected.

**Proof.** Using the Transitiviy Axiom, all the possible outcomes of all the alternatives can be ranked in terms of preferability. Suppose the outcomes are so labeled that $c_1$ is preferred to $c_2$, and $c_2$ is preferred to $c_3$, and so on. The most preferred outcome is $c_1$ while the least preferred one is $c_n$. For the Reduction Axiom, each alternative $A$ has an equivalent alternative which can be expressed in terms of all outcomes and their associated probabilities $P(c_1|A), P(c_2|A), \ldots, P(c_n|A)$. For instance, if alternative $A$ has only the outcomes $c_3$ and $c_7$ then $A$ is equivalent to an alternative with outcomes $c_1, \ldots, c_n$ where all the probabilities
with \( c_i \) different from \( c_i \) and \( c_T \) are zero. Suppose to have two alternatives \( A \) and \( B \), express \( A \) and \( B \) in function of all the possible ranked outcomes \( c_1, c_2, \ldots, c_n \). For the Continuity Axiom there is a number \( u(c_i) \) such that \( c_i \) is equally preferred to an alternative with probability \( u(c_i) \) of yielding \( c_1 \) and probability \( 1 - u(c_i) \) of yielding \( c_n \). Thus, by the Substitution Axiom, the alternatives \( A \) and \( B \) are equally preferred to the alternatives \( \tilde{A} \) and \( \tilde{B} \) where each outcome \( c_i \) is substituted with a binary gamble with probability \( u(c_i) \) of yielding \( c_1 \) and a probability \( 1 - u(c_i) \) of yielding \( c_n \). By the Reduction Axiom, \( \tilde{A} \) is equally preferred to an alternative with probability \( \sum_{i=1}^{n} p(c_i|A)u(c_i) \) of yielding \( c_1 \) and a probability \( 1 - \sum_{i=1}^{n} p(c_i|A)u(c_i) \) of yielding \( c_n \). Similarly, \( \tilde{B} \) is equally preferred to an alternative with a probability \( \sum_{i=1}^{n} p(c_i|B)u(c_i) \) of yielding \( c_1 \) and a probability \( 1 - \sum_{i=1}^{n} p(c_i|B)u(c_i) \) of yielding \( c_n \). Thus, by the Substitution Axiom, \( \tilde{A} \) and \( \tilde{B} \) can be replaced by these alternatives which have only outcomes \( c_1 \) and \( c_n \). By the Monotonicity Axiom, \( A \) is preferred to \( B \) only if the probability of yielding \( c_1 \) is higher for \( A \) than for \( B \), or:

\[
\sum_{i=1}^{n} p(c_i|A)u(c_i) > \sum_{i=1}^{n} p(c_i|B)u(c_i).
\]

The function \( u(c_i) \) is the utility function of the decision maker, this function associates to each level of wealth \( c_i \) a utility value between 0 and 1.

### A.3.2 Properties of utility functions

There are some general properties that hold for utility functions. If one assumes that more is always preferred to less, the utility function will be strictly monotonically increasing and its first derivative will be strictly positive. Concerning the second derivative of a utility function, the following property holds.

**Property A.3.1.** If risk aversion holds, then the utility function \( u(x) \) is concave, i.e. \( u''(x) < 0 \). Risk neutrality implies that \( u(x) \) is linear, i.e. \( u''(x) = 0 \). Risk seeking behavior implies that the utility function is convex, i.e. \( u''(x) > 0 \).

**Proof.** Consider an alternative with the possible outcomes \( \bar{x} - \epsilon \) and \( \bar{x} + \epsilon \) which are equally preferred. If the decision maker is risk averse, then this alternative must be less preferred than receiving the expected value \( \bar{x} \) of the decision. Thus

\[
u(\bar{x}) > \frac{1}{2}[u(\bar{x} - \epsilon) + u(\bar{x} + \epsilon)].\]

If we write the right hand of the above expression as a Taylor expansion around \( \bar{x} \) we obtain:

\[
u(\bar{x}) > \frac{1}{2}[u(\bar{x}) - \frac{du(\bar{x})}{dx} \epsilon + \frac{1}{2} \frac{d^2u(\bar{x})}{dx^2} \epsilon^2 + \cdots + u(\bar{x}) + \frac{du(\bar{x})}{dx} \epsilon + \frac{1}{2} \frac{d^2u(\bar{x})}{dx^2} \epsilon^2 + \cdots].\]
If $\epsilon \to 0$ we obtain:

$$u(x) > u(x) + \frac{1}{2} \frac{d^2 u(x)}{dx^2} \epsilon^2,$$

this is equivalent to:

$$\frac{1}{2} \frac{d^2 u(x)}{dx^2} \epsilon^2 < 0,$$

which holds only if:

$$\frac{d^2 u(x)}{dx^2} < 0.$$

As $x$ can be any value where the utility function is defined, the result in the case of risk aversion is established. For risk-neutral or risk-seeking behavior the result is obtained considering that $u(x) = \frac{1}{2} [u(x-\epsilon)+u(x+\epsilon)]$ in the case of neutrality and $u(x) < \frac{1}{2} [u(x-\epsilon)+u(x+\epsilon)]$ in the case of risk seeking behavior.

**Definition A.3.1 (Certain equivalent).** The certain equivalent of an alternative is defined as the amount $CE$ such that the decision maker is indifferent between the alternative and its certain equivalent.

Based on the above definition, one can state the following property.

**Property A.3.2.** For any utility function $u(x)$, given an alternative and its certain equivalent $CE$ then:

$$u(CE) = E[u(x)].$$

**Proof.** The alternative is equally preferred to the certain equivalent $CE$. Therefore the expected utility of the two choices must be the same:

$$E[u(x)] = 1 \cdot u(CE) = u(CE),$$

where $u(CE)$ is the expected utility of a gamble in which the outcome $CE$ has probability 1. \(\square\)

The above property can be used to find the certain equivalent of an alternative if the utility function is known. Since $u(x)' > 0$ the utility function is invertible and thus $CE = u^{-1}(E[u(x)])$.

**Theorem A.3.2.** Two utility functions $u(x)$ and $\tilde{u}(x)$ are guaranteed to give the same ranking if and only if:

$$\tilde{u}(x) = au(x) + b$$

for some constant $a > 0$ and $b$. 
Proof. If alternative $A$ is preferred to $B$ according to $u(x)$ then:

$$E_A[u(x)] > E_B[u(x)].$$

Assuming that $\tilde{u}(x) = au(x) + b$, then $E[\tilde{u}(x)] = aE[u(x)] + b$, and

$$E[u(x)] = \frac{E[\tilde{u}(x)] - b}{a}. $$

Since $A$ is preferred to $B$, then:

$$\frac{E_A[\tilde{u}(x)] - b}{a} > \frac{E_B[\tilde{u}(x)] - b}{a}$$

consequently, for $a > 0$

$$E_A[\tilde{u}(x)] > E_B[\tilde{u}(x)]$$

thus, the utility function $\tilde{u}(x)$ preserves the ranking of the alternatives. If we do not make any assumption about $\tilde{u}(x)$, we find that:

$$E_A[u(x)] > E_B[u(x)] \rightarrow aE_A[u(x)] + b > aE_B[u(x)] + b$$

$$\rightarrow E_A[au(x) + b] > E_B[au(x) + b].$$

Therefore a function leading to the same ranking of alternatives is $\tilde{u}(x) = au(x) + b$.   

**Definition A.3.2 (Strategic equivalence).** Two utility functions that imply the same preference ranking for any two lotteries are said to be strategic equivalent.

**Definition A.3.3 (Local risk aversion).** The local risk aversion at point $x$, written as $r(x)$ is defined by

$$r(x) = \frac{-u''(x)}{u'(x)}.$$

It is easy to verify that two utility functions are strategically equivalent if and only if they have the same local risk-aversion function [43]. Moreover, if $r(x)$ is always positive, then the decision maker is risk averse since the utility function is concave and monotonically increasing. If $r$ is always negative then the decision maker is risk seeking. If $r$ is zero, then the utility function is linear since the second derivative is zero. A linear utility function of the form $ax + b$ with $a > 0$ ($u'(x) > 0$) induces the same ranking as the expected value. Therefore if $r$ is zero the decision maker is risk neutral.

**Definition A.3.4 (Constant risk aversion).** A decision maker is constantly risk averse if the local risk aversion $r(x)$ is a positive constant.

**Definition A.3.5 (Constant risk proneness).** A decision maker is constantly risk prone if the local risk aversion $r(x)$ is a negative constant.
Theorem A.3.3 (Exponential utility). An investor is constantly risk averse or constantly risk prone if and only if his utility function is an exponential of the form:

\[ u(x) = a - be^{-cx}, \]

where \( b > 0, c > 0 \) in the case of risk aversion and \( b < 0, c < 0 \) in the case of risk proneness.

Proof. By solving the differential equation \( r(x) = c \) with \( c \neq 0 \) and imposing \( u'(x) > 0 \) we obtain the above exponential utility function. Assuming an exponential utility function we obtain a constant value of \( r(x) = c \).

Theorem A.3.4 (Linear utility). An investor is risk neutral if and only if his utility function is linear:

\[ u(x) = a + bx, \]

where \( b > 0 \) if the utility function is monotonically increasing.

Proof. By solving the differential equation \( r(x) = 0 \) and imposing \( u'(x) > 0 \) we obtain the above linear function. Assuming a linear utility function we obtain a value of \( r(x) = 0 \).

A.3.3 Estimation of the utility function

If one assumes that the decision maker is constantly risk averse (this is often the case), then the utility function must be of the form:

\[ u(x) = a - be^{-cx}, \]

where \( c \neq 0 \) is the risk-aversion coefficient and \( b \) is positive if the utility function is monotonically increasing. As \( c \) gets larger, the decision maker becomes more averse to risk, it can be shown that if \( c \to 0 \) the utility tends to be risk neutral. The usual convention is to scale the utility function so that the least preferred level (Low) of the evaluation measure has a utility of zero and the most preferred (High) has a utility of one. With this convention the utility function can be written as:

\[ u(x) = \frac{e^{-(x-Low)c} - 1}{e^{-(High-Low)c} - 1}. \]

In order to estimate the risk-aversion coefficient \( c \) we solve the equation:

\[ E[u(x)] = u(CE). \]

This implies that it is necessary to find the certain equivalent for a given gamble. For example, if the alternative has equal chances of Low and High and the certainty equivalent is \( CE \), then we obtain:

\[ E[u(x)] = \frac{1}{2}u(High) + \frac{1}{2}u(Low) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 = 0.5, \]
therefore
\[ u(CE) = e^{-(CE-Low)c - 1} e^{-(High-Low)c - 1} = 0.5. \]
By solving numerically this equation we obtain the value of \( c \) and therefore an estimation of the investor’s utility function.

There are other two methods for estimating the utility function of the decision maker. These methods do not make assumptions on the analytical form that the utility function should have, they are in fact based on the punctual estimation of the utility values:

- **Certainty equivalent method.** Given a gamble with probability \( p_1 \) of outcome \( High \) and probability \( 1 - p_1 \) of outcome \( Low \), and the certainty equivalent for that gamble is \( CE_1 \) we obtain:

  \[ E[u(x)] = u(CE_1) = p_1 \cdot u(High) + (1 - p_1) \cdot u(Low) = p_1. \]

  Now we imagine to have a second gamble with probability \( p_2 \) of outcome \( CE_1 \) and probability \( 1 - p_2 \) of outcome \( Low \), and the certainty equivalent for that gamble is \( CE_2 \) we obtain:

  \[ E[u(x)] = u(CE_2) = p_2 \cdot u(CE_1) + (1 - p_2) \cdot u(Low) = p_1 \cdot p_2. \]

  If we iterate the process we obtain the punctual estimates:

  \[ u(CE_1), u(CE_2), \ldots, u(CE_n), \]

  with \( Low \leq CE_i \leq High \). In this way we obtain a punctual estimation of the utility function.

- **Lottery equivalent method.** This method differs from the previous one because the certainty equivalent is not known here. What is known is the equivalent lottery. Suppose there is a gamble with probability \( p \) of outcome \( High \) and probability \( 1 - p \) of outcome \( Low \). Suppose now that this gamble is equivalent (for the decision maker) to a gamble with probability \( \tilde{p} \) of outcome \( x \) and probability \( 1 - \tilde{p} \) of outcome \( Low \). Since the two gambles are equally preferred, they must have the same expected utilities:

  \[ E[u(x)]_{\text{gamble1}} = E[u(x)]_{\text{gamble2}}, \]

  thus:

  \[ p \cdot u(High) + (1 - p) \cdot u(Low) = \tilde{p} \cdot u(x) + (1 - \tilde{p}) \cdot u(Low). \]

  By solving the above equation we obtain:

  \[ u(x) = \frac{p}{\tilde{p}}. \]

  Now the process can be iterated with different values of \( x \) and by exploiting the utility values found previously.
A.3. Utility theory

These two methods can be combined, the basic idea is always the same: if two alternatives are equally preferred then they must have the same expected utility. In practice, a questionnaire needs to be developed from which it is possible to estimate the decision maker’s utility function by asking domain-specific questions.

A.3.4 Portfolio selection using utility functions

According to the Expected Utility Theorem, the optimal portfolio is the one which maximizes the expected utility of the decision maker. Assuming that the portfolio is given by a linear combination of assets, the optimization problem is the following:

\[
\max_x \ E[u(v)]
\]

\[v = x^T y\]

\[q^T x = \beta\]  \hspace{1cm} (A.17)

where \(v\) is the value of the portfolio, \(u(\cdot)\) is the utility function, \(x\) is the decision vector, \(\beta\) is the budget, \(y\) are assets values and \(q\) are the assets costs.

In order to reconcile the utility approach with the risk management approach, and in particular with the Markowitz framework, we can Taylor expand the utility function around the mean value of the portfolio \(\mu_v\) until the fourth order:

\[u(v) \simeq u(\mu_v) + u'(\mu_v)(v - \mu_v) + \frac{1}{2!} u''(\mu_v)(v - \mu_v)^2 + \frac{1}{3!} u'''(\mu_v)(v - \mu_v)^3 + \frac{1}{4!} u''''(\mu_v)(v - \mu_v)^4.\]  \hspace{1cm} (A.18)

By applying the mean operator to the left and the right hands we obtain:

\[E[u(v)] \simeq u(\mu_v) + \frac{1}{2} u''(\mu_v) E[(v - \mu_v)^2] + \frac{1}{3!} u'''(\mu_v) E[(v - \mu_v)^3] + \frac{1}{4!} u''''(\mu_v) E[(v - \mu_v)^4].\]  \hspace{1cm} (A.19)

Now we define the following risk tolerance parameters:

\[\alpha \triangleq \frac{1}{2} u''(\mu_v),\]

\[\beta \triangleq \frac{1}{6} \sigma_v^3 u'''(\mu_v),\]

\[\gamma \triangleq \frac{1}{24} \sigma_v^4 u''''(\mu_v).\]
The expected utility can be written in function of the variance, the skewness and the kurtosis of the value distribution:

\[ E[u(v)] \simeq u(\mu_v) + \alpha \cdot \sigma_v^2 + \beta \cdot Skew(v) + \gamma \cdot Kurt(v). \]

The skewness and the kurtosis are the standardized third and fourth central moments of the distribution, defined respectively as:

\[ Skew(v) = \frac{(v - \mu_v)^3}{\sigma_v^3}, \]
\[ Kurt(v) = \frac{(v - \mu_v)^4}{\sigma_v^4}. \]

The skewness is a measure of symmetry of the distribution. If a probability density function is asymmetric towards the right the skewness is positive, if it is symmetric the skewness is zero and if it is asymmetric towards the left the skewness is negative. Often the kurtosis of a distribution is compared to the kurtosis of a normal distribution which is equal to 3. A kurtosis larger than 3 indicates long flat tails, if the kurtosis is smaller than 3 there are peaks.

Note that \( \alpha \) is negative if the investor is risk averse, since \( u(y)'' < 0 \) in this case. If the skewness and the kurtosis are not considered we obtain the Markowitz framework expressed in the Lagrangian form (A.7) on page 125. In fact, the utility function is monotonically increasing and maximizing \( u(\mu_v) \) is equivalent to maximize \( \mu_v \). In the following two cases the utility framework is equal to the mean-variance framework.

If the underlying assets are normally distributed, then the value of the portfolio is normally distributed. As normal distributions are symmetric, the skewness is zero, moreover, the kurtosis has a constant value of 3. Therefore we can drop the skewness from the expected utility. The kurtosis is constant and the risk coefficient \( \gamma \) depends only on the variance. Therefore for a given expected value, the variance is minimized if \( \gamma \leq 0 \). This is the case for a constantly risk averse decision maker. This consideration holds also considering the entire Tailor expansion of the utility, the standardized central moments of higher orders are null or constant for the normal distribution.

The second case is when the utility function is quadratic. In fact, all the coefficients in the Taylor expansion of order \( \geq 3 \) are null, therefore only the mean and the variance are left.

In general, risk-averse decision makers prefer positively skewed portfolios [28] since such portfolios have larger probabilities of high outcomes. Moreover, it has been shown that portfolio optimization based on the maximization of expected value and the minimization of variance can underestimate the risk if the value distribution is not symmetric [53]. The higher moments of the distributions should therefore be included in the process of selecting an optimal portfolio in this case.
A.4 Conclusions

Risk management and utility theory provide the analytic tools to reduce risk by diversification and to evaluate the risk profile of the decision maker. The most used risk measures in practice are the standard deviation, the Value-at-Risk, and the Conditional Value-at-Risk. While the standard deviation (and the variance) is appropriate if the probability density of the loss function is symmetric around the mean, the Value-at-Risk and the Conditional Value-at-Risk explicitly take into account the downside risk. Thus, they are more appropriate if the loss distribution is not symmetric.

Under the assumption of normally distributed assets and of constant absolute risk aversion, expected utility theory leads to the classical mean-variance framework. This framework prescribes the minimization of the risk as measured by the variance or the standard deviation, while imposing a minimum level of expected value.
Bibliography


Curriculum Vitae

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2003 - 2004 PhD, University of St. Gallen (HSG)
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